Opportunities for Systemic Improvement in K-16 Mathematics Teaching, Learning, Curriculum and Assessment

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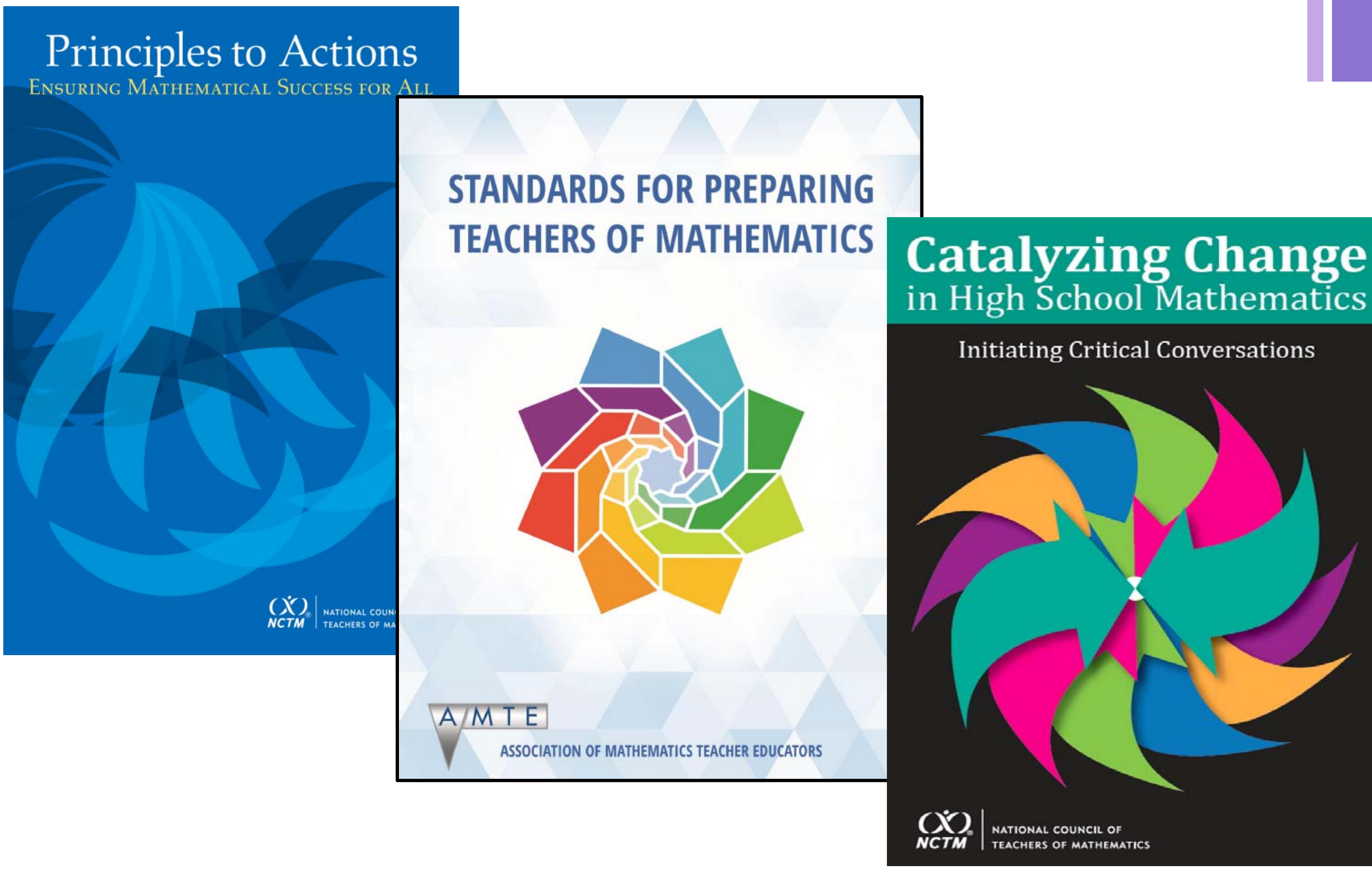
May 15, 2019

Goals

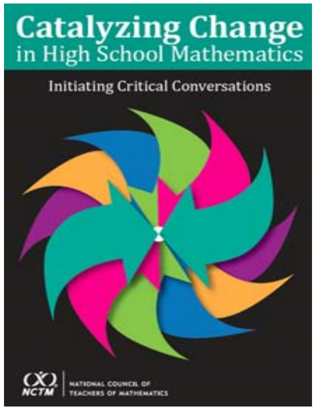
Examine recent recommendations from mathematics, mathematics education, and other organizations to improve mathematics curriculum, teaching and learning in:

- Your department/university;
- K-12 districts in which you work;
- Your own practices.

Recent Recommendations



Key Recommendations



- Each and every student should learn the Essential Concepts in order to expand professional opportunities, understand and critique the world, and experience the joy, wonder, and beauty of mathematics.
- High school mathematics should discontinue the practice of tracking teachers as well as the practice of tracking students into qualitatively different or dead-end course pathways.
- Classroom instruction should be consistent with research-informed and equitable teaching practices.
- High schools should offer continuous four-year mathematics pathways with all students studying mathematics each year, including two to three years of mathematics in a common shared pathway focusing on the Essential Concepts, to ensure the highest-quality mathematics education for all students.

Conference Board of the Mathematical Sciences

AMATYC American Mathematical Association of Two-Year Colleges
AMS American Mathematical Society
ASA American Statistical Association
ASL Association for Symbolic Logic
AWM Association for Women in Mathematics
AMTE Association of Mathematics Teacher Educators
ASSM Association of State Supervisors of Mathematics
BBA Benjamin Banneker Association
INFORMS Institute for Operations Research and the Management Sciences
IMS Institute of Mathematical Statistics
MAA Mathematical Association of America
NAM National Association of Mathematicians
NCSM Leadership in Mathematics Education
NCTM National Council of Teachers of Mathematics
SIAM Society for Industrial and Applied Mathematics
SOA Society of Actuaries
TODOS: Mathematics for All
WME Women and Mathematics Education

CBMS Forum, May 5-7, 2019

High School to College Mathematics Pathways: Preparing Students for the Future

Goal:

To help state leadership teams ***draw on the expertise*** of the CBMS societies and the Dana Center to ***form task forces*** working to ***coordinate efforts across grades 11-14*** that will lead states to ***create policies and practices*** for mathematics instruction that contribute to ***successful completion without reducing quality***.

CBMS Forum, May 5-7, 2019
High School to College Mathematics Pathways: Preparing Students for the Future

Alabama

Arizona

Aransas

California

Florida

Idaho

Indiana

Iowa

Kansas

Maine

Maryland

Massachusetts

Minnesota

Missouri

Nebraska

Ohio

Oklahoma

Oregon

Tennessee

Utah

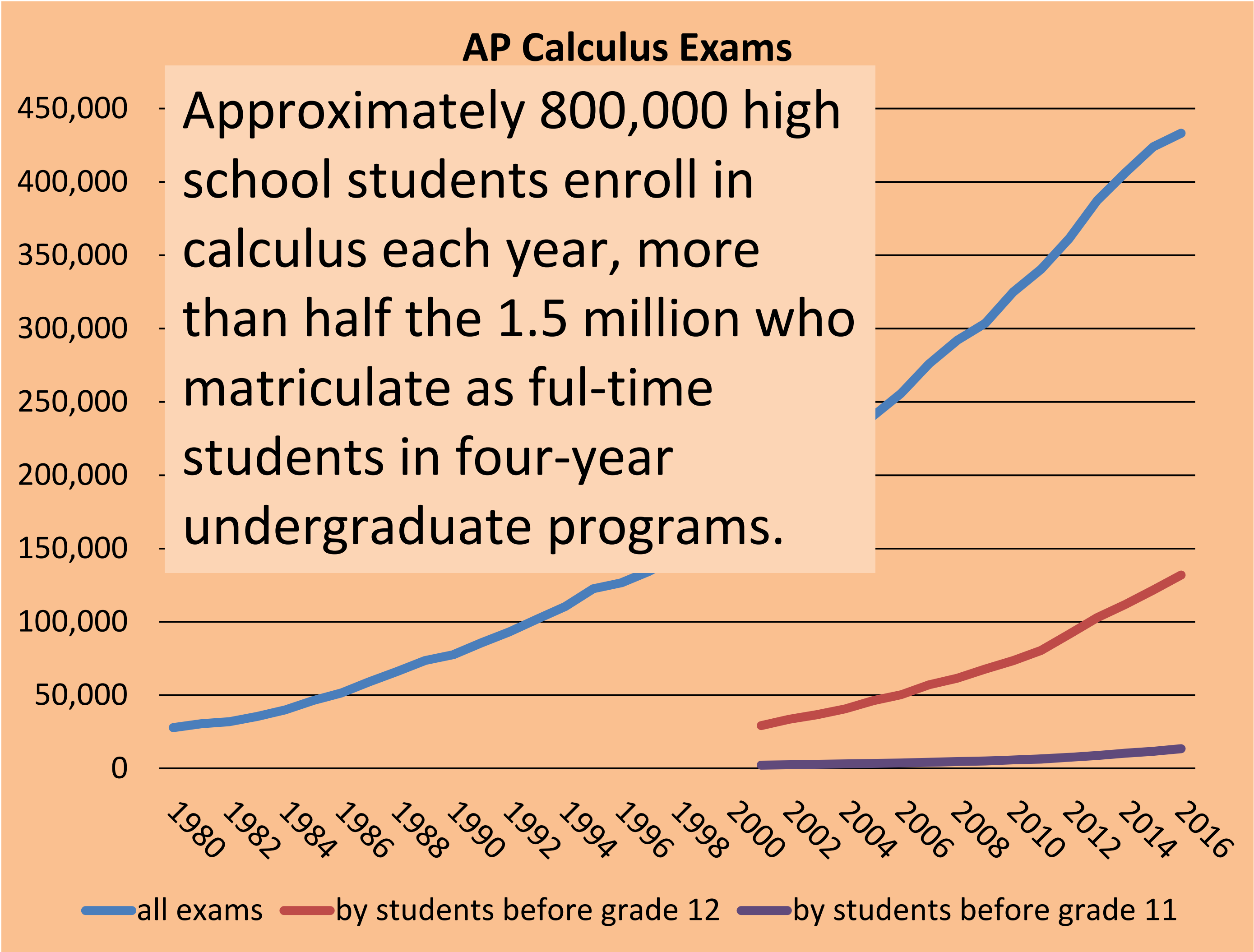
Washington

Wisconsin

Themes

- Policies and practices that support success of each and every student (equity)
- Need to update the mathematics curriculum
  - Modeling
  - Statistics
  - Mathematical processes/SMPs
- Effective mathematics instructional practices

AP Calculus Exam-Taking is Increasing



Source: Bressoud. (2017). The Role of Calculus in the Transition from High School to College Mathematics.

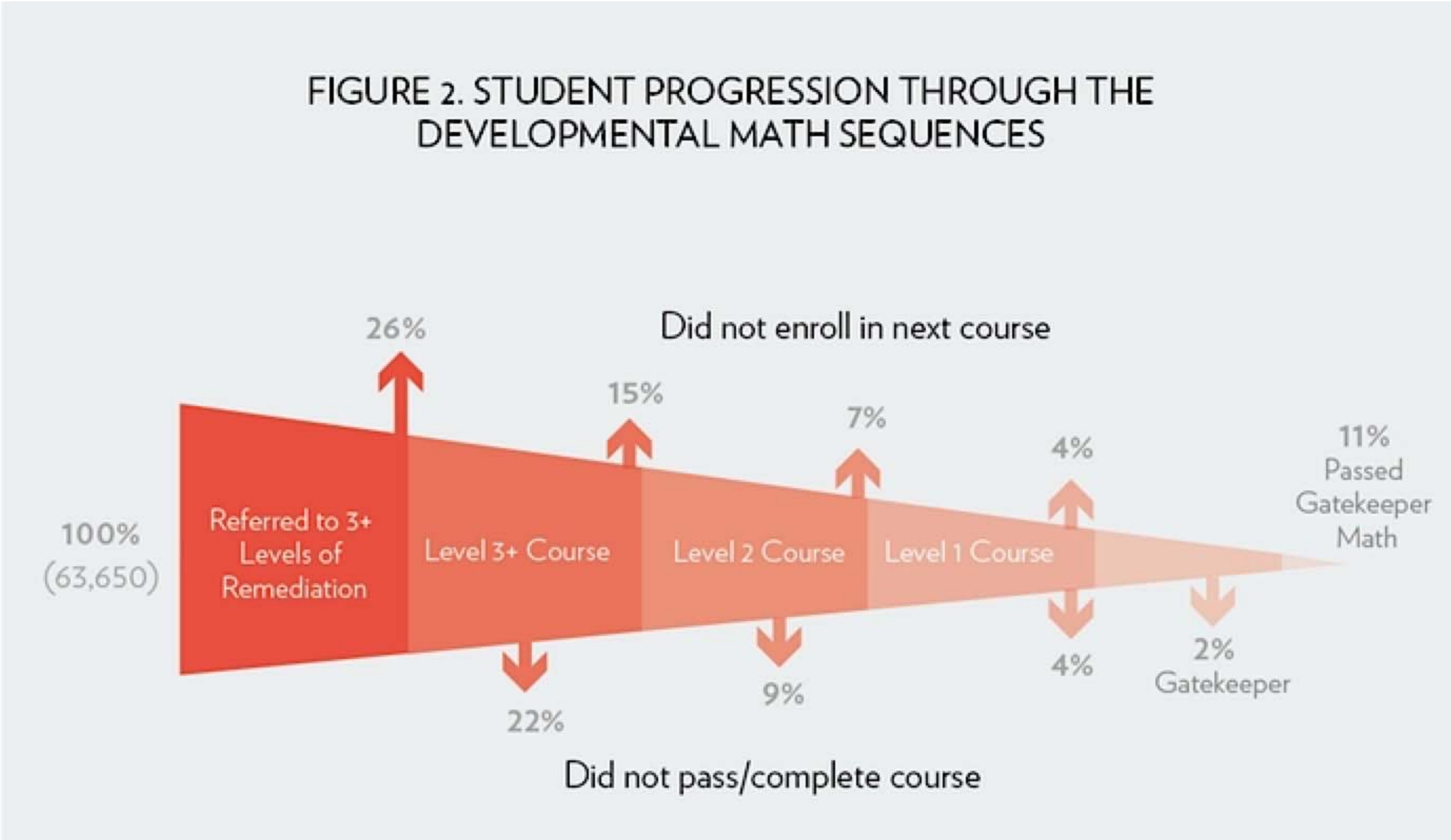
Most Calculus I Students Took Calculus in High School

Course taking in High School By students in Calculus I at PhD Universities		
	percentage	Mean grade (SD)
Algebra II* $\leq 10^{th}$ grade	77%	3.8 (0.5)
Precalculus* $\leq 11^{th}$ grade	67%	3.7 (0.6)
Calculus $\leq 12^{th}$ grade	67%	3.8 (0.5)
Statistics $\leq 12^{th}$ grade	9%	3.6 (0.6)

* Does not count students who took an integrated curriculum

Source: Bressoud. (2017). The Role of Calculus in the Transition from High School to College Mathematics.

The Need For Reform



Source: "What We Know about Developmental Education Outcomes," CCRC, Teachers College, Columbia University, January 2014, 5, http://ccrc.tc.columbia.edu/media/k2/attachments/what-we-know-about-developmental-education-outcomes.pdf

Dana Center Principles for Pathways

Quick structural change

Mathematics pathways are structured so that:

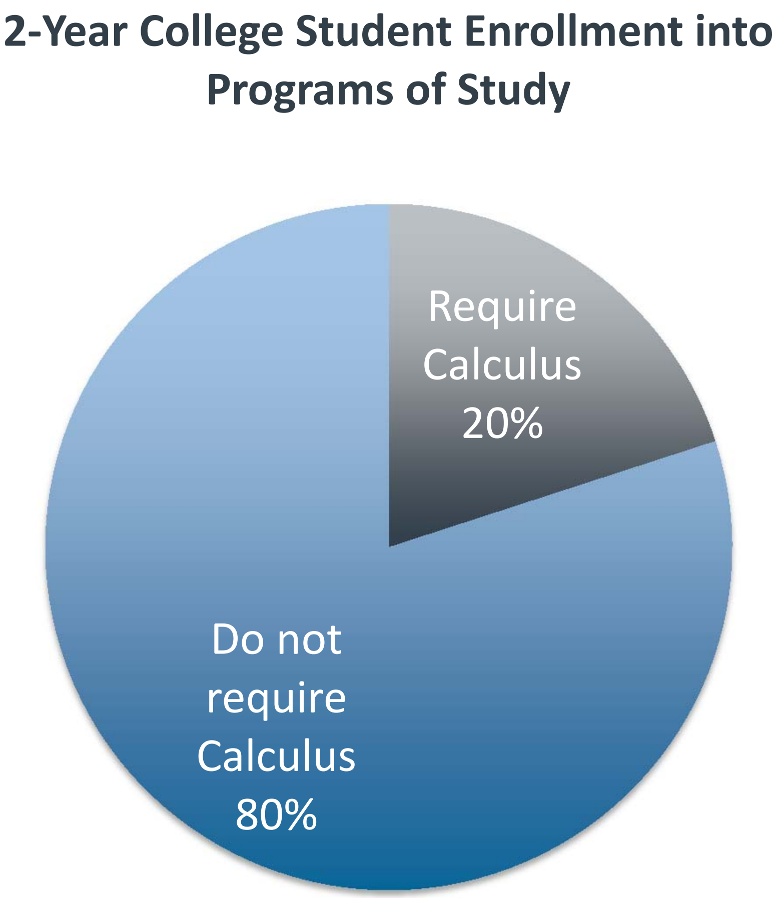
1) All students, regardless of college readiness, enter directly into mathematics pathways aligned to their programs of study.
2) Students complete their first college-level math requirement in their first year of college.

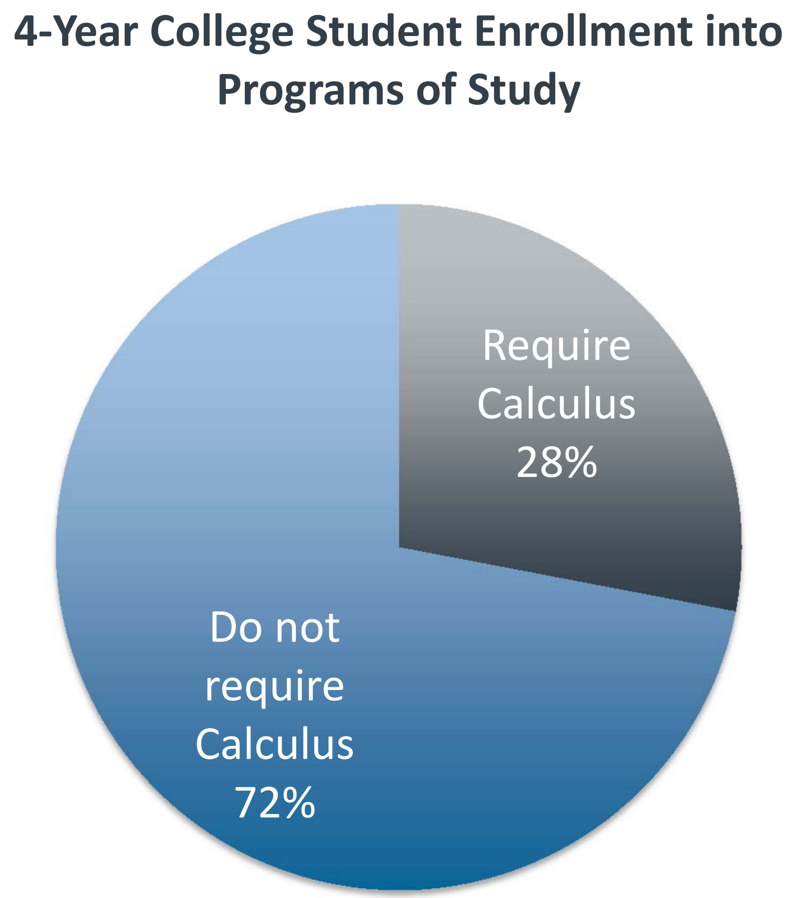
Continuous improvement

Students engage in a high-quality learning experience in math pathways designed so that:

3) Co-requisite strategies to support students as learners are integrated into courses and are aligned across the institution.
4) Instruction incorporates evidence-based curriculum and pedagogy.

Pathways aligned placement





Burdman, P. (2015). *Degrees of freedom: Diversifying math requirements for college readiness and graduation.* Oakland CA: Learning Works and Policy Analysis for California Education.

States Implementing Co-Requisite Math at Scale

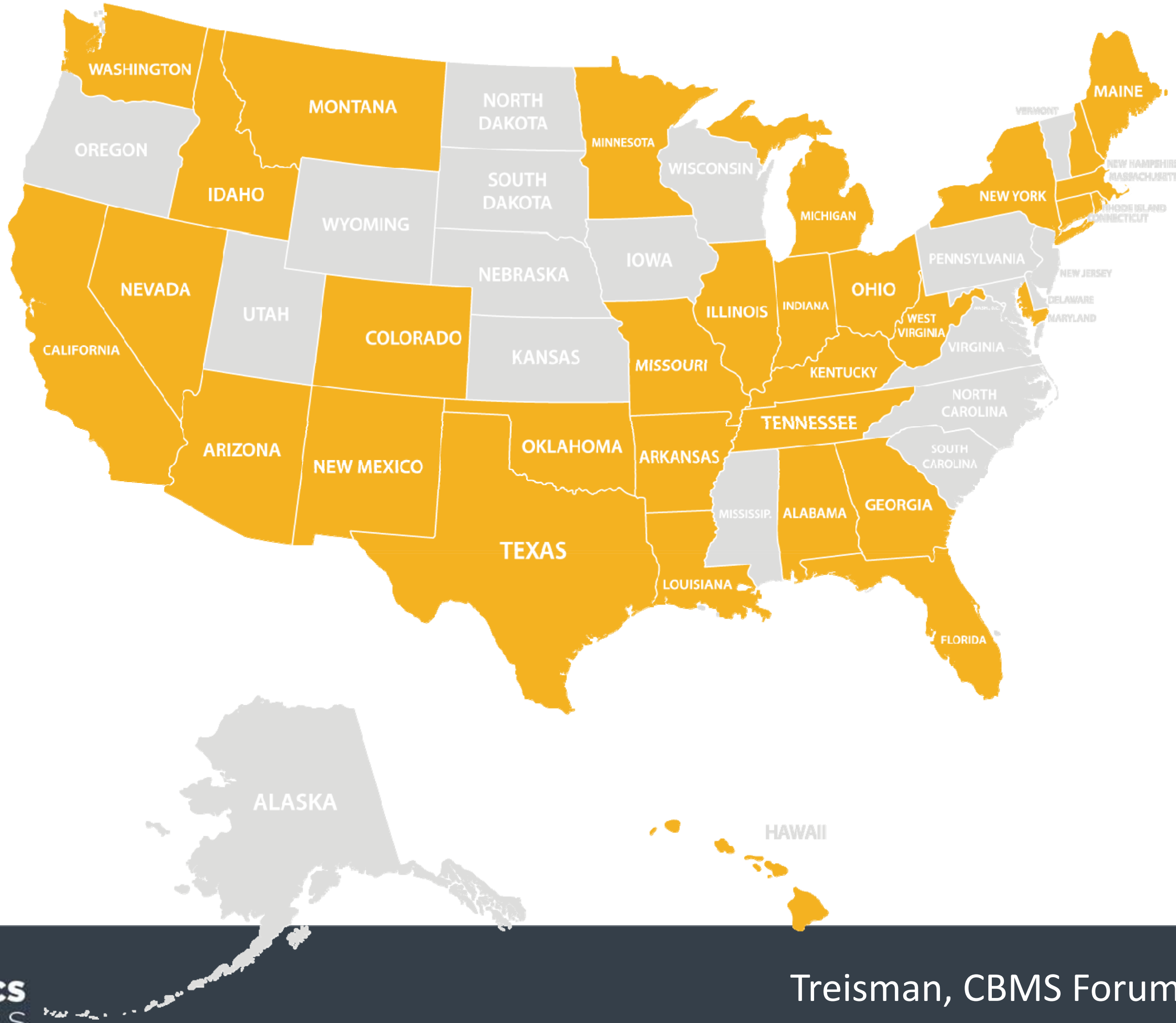
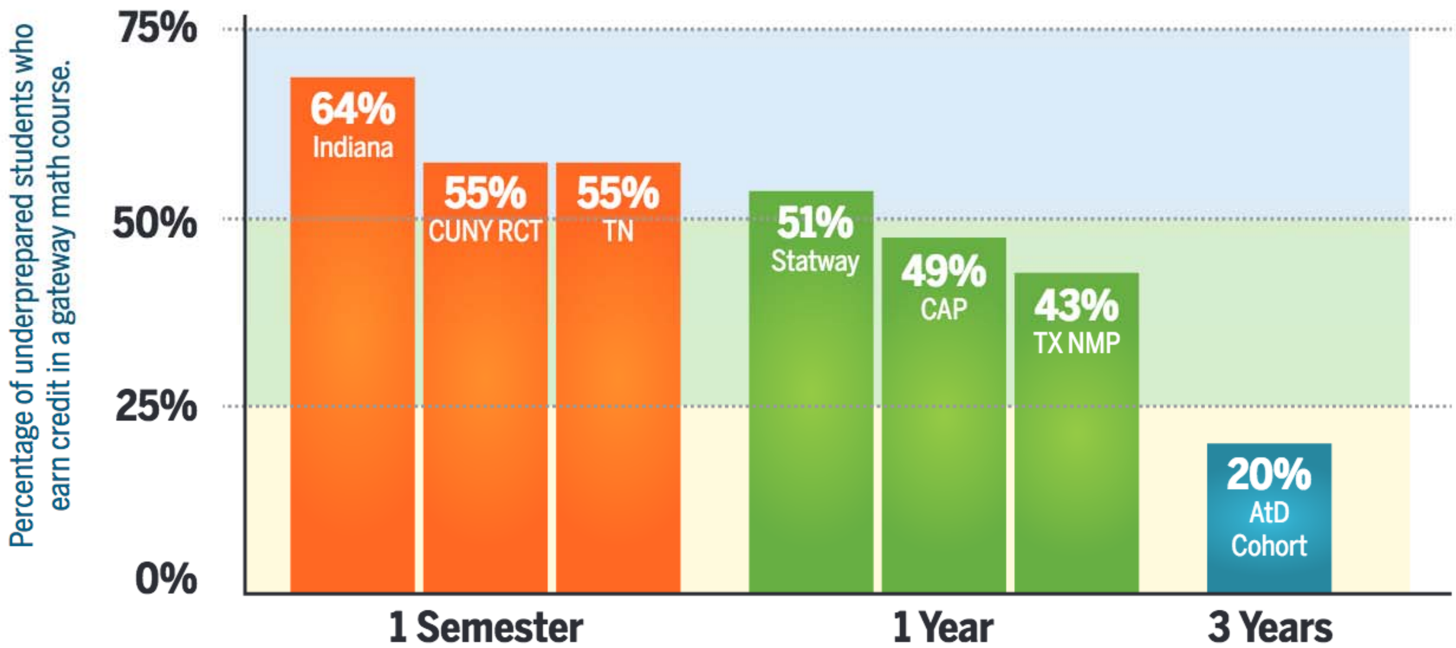


Figure 2. A Preponderance of Evidence
More students succeed in less time with accelerated models

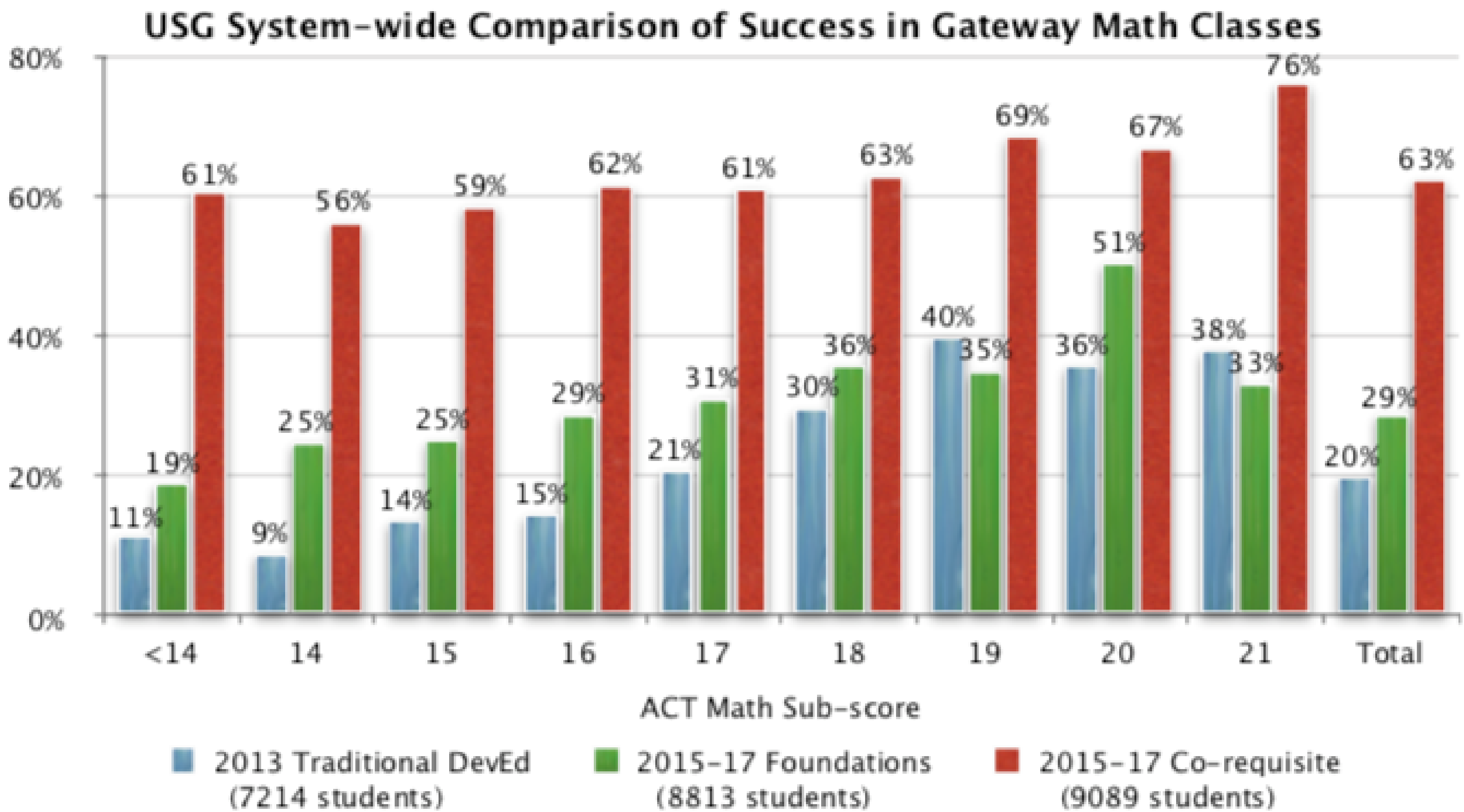


Consistent results across multiple sites using different models show that more students earn credit in less time with accelerated models.

Sources: Indiana (Complete College America, 2016); CUNY (Logue et al., 2016); Tennesse (Tennessee Board of Regents, 2016); Statway (Sowers & Yamada, 2015); CAP (California Acceleration Project, 2015); TX NMP (Rutschow & Diamond, 2015); and AtD (Bailey et al., 2010).

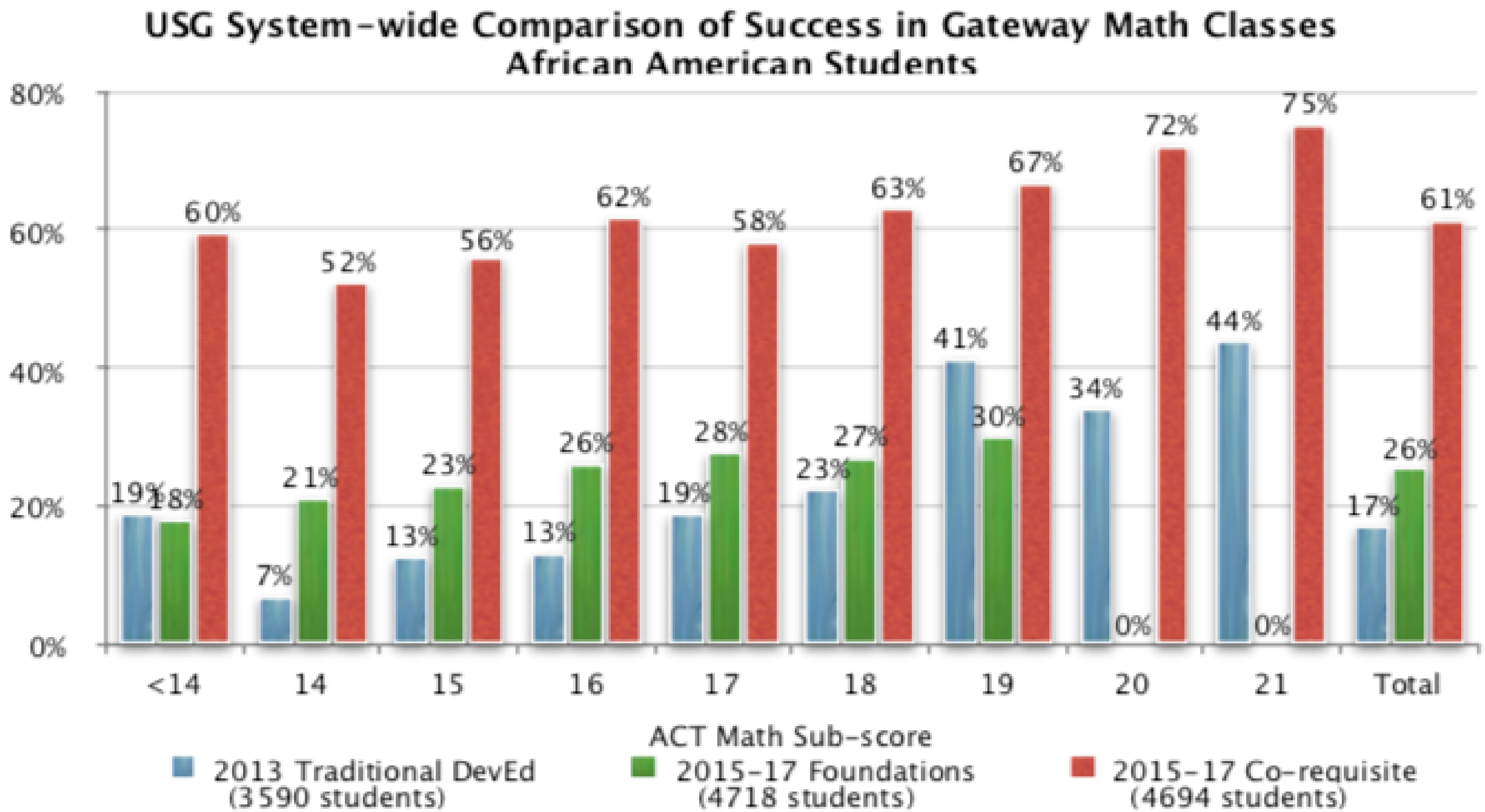
DCMP (2019). The Case for Mathematics Pathways

Georgia Co-Requisite Model



Source: Denley, T., *CoRequisite Developmental Mathematics*

Georgia Co-Requisite Model



Source: Denley, T., *CoRequisite Developmental Mathematics*

Systemic dimensions of math pathway reforms

Student-centered

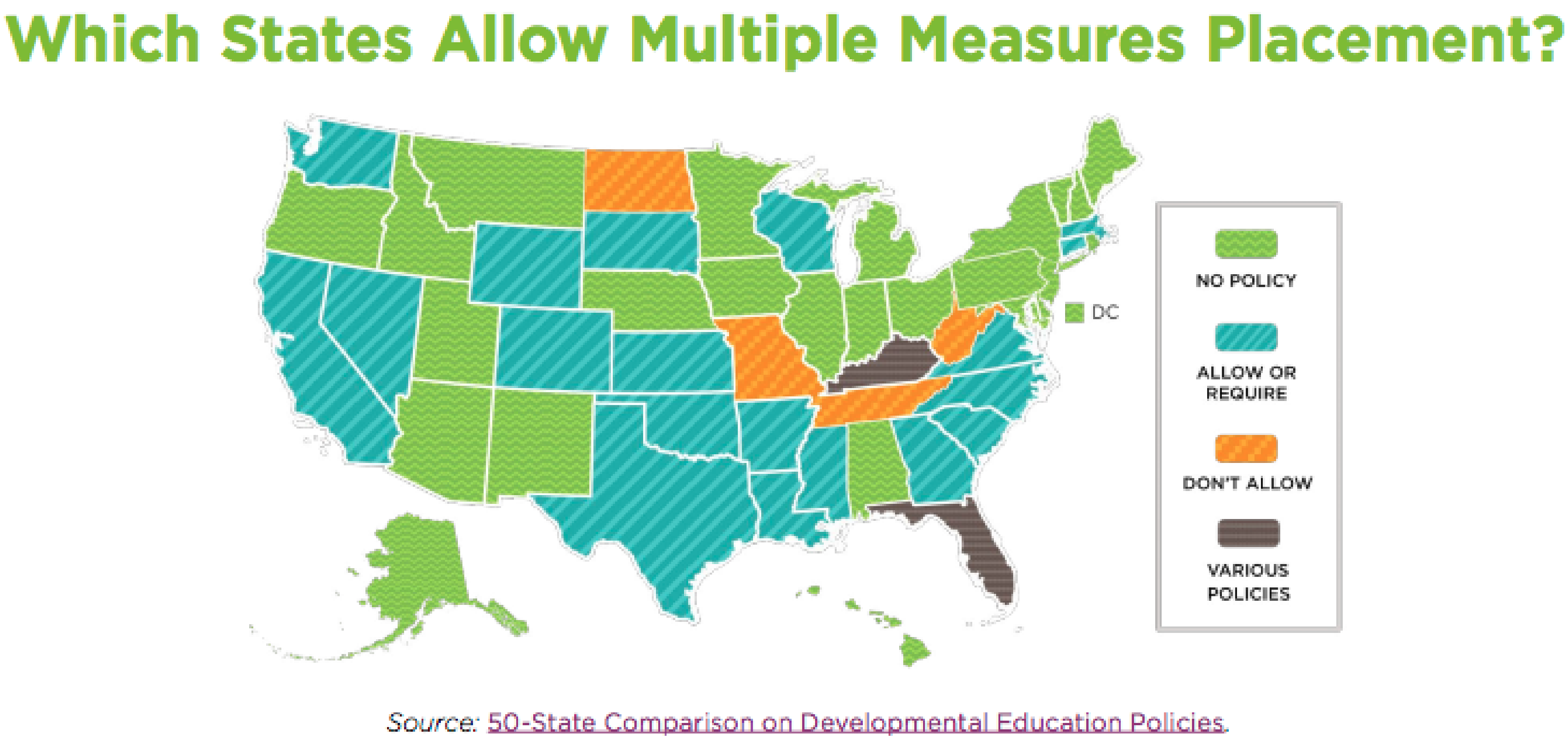
Faculty-driven

Administrator-supported

Policy-enabled

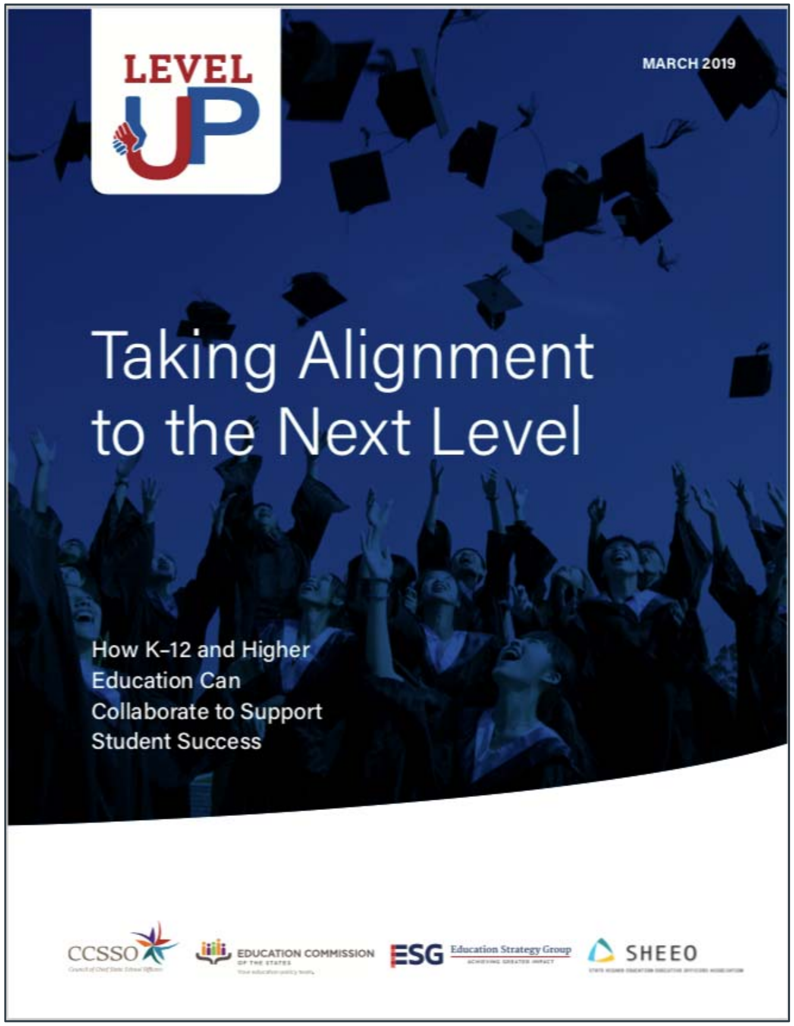
Culturally-reinforced

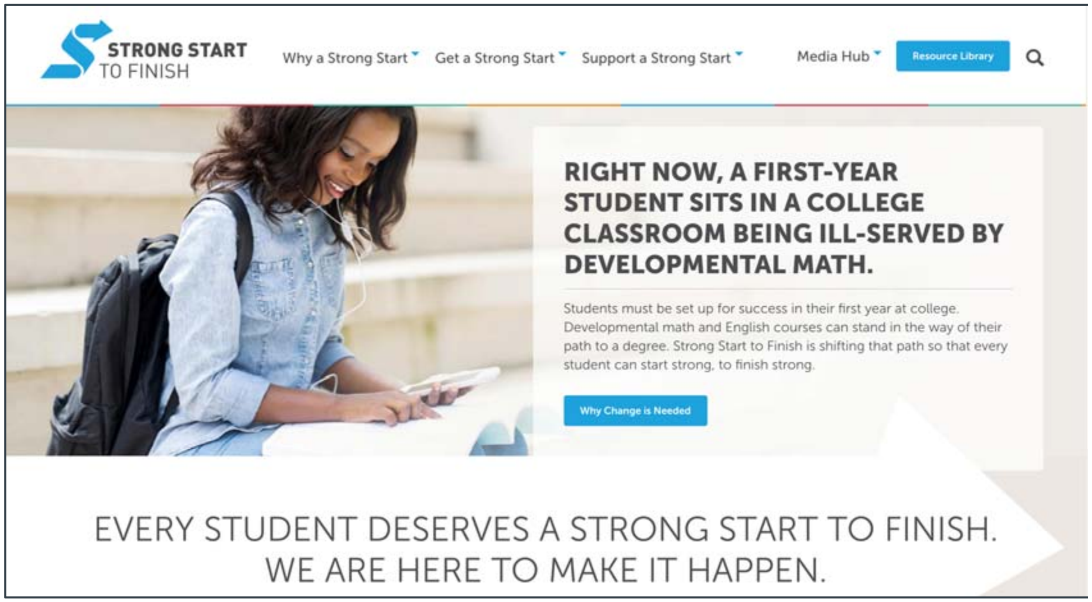
States using high school data for college placement



Source: ECS, 2019

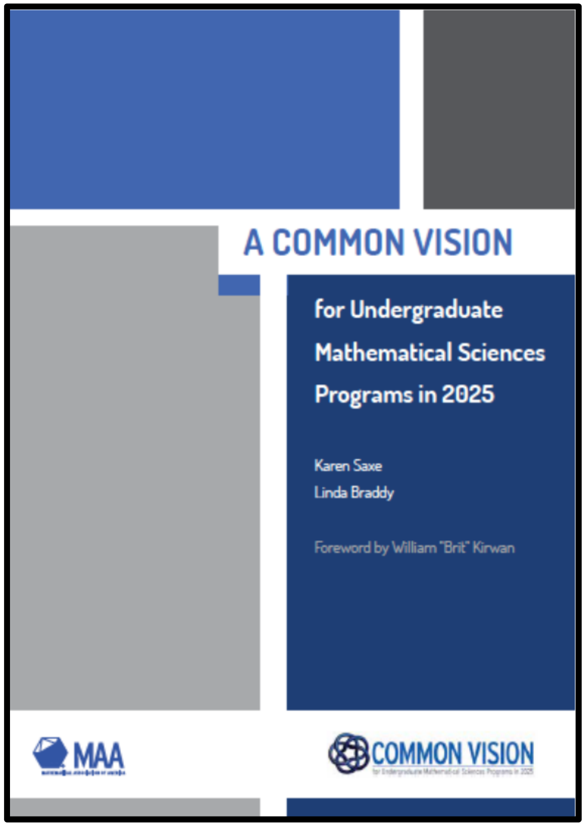
Supporting seamless transitions





+ A Common Vision for Undergraduate Mathematical Sciences in 2025

"The status quo is unacceptable."



Calls on the community to:

1. Update curricula
2. Articulate clear pathways between curricula driven by changes in K-12 and the first courses students take in college
3. Scale up the use of evidence-based pedagogical methods

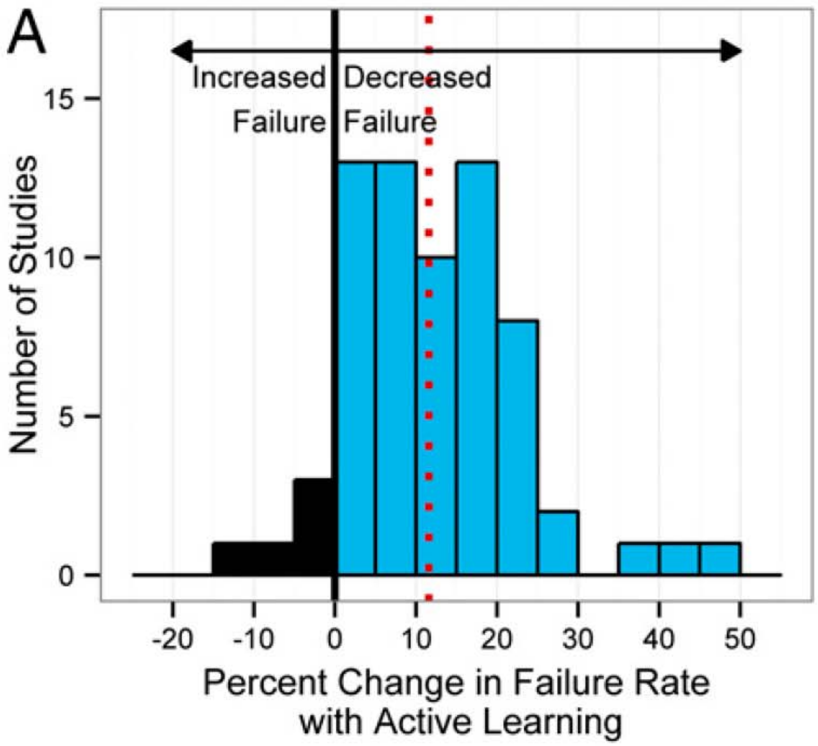
AMATYC   AMS   ASA   MAA   siam

Active Learning Increases Student Performance in Science, Engineering, and Mathematics

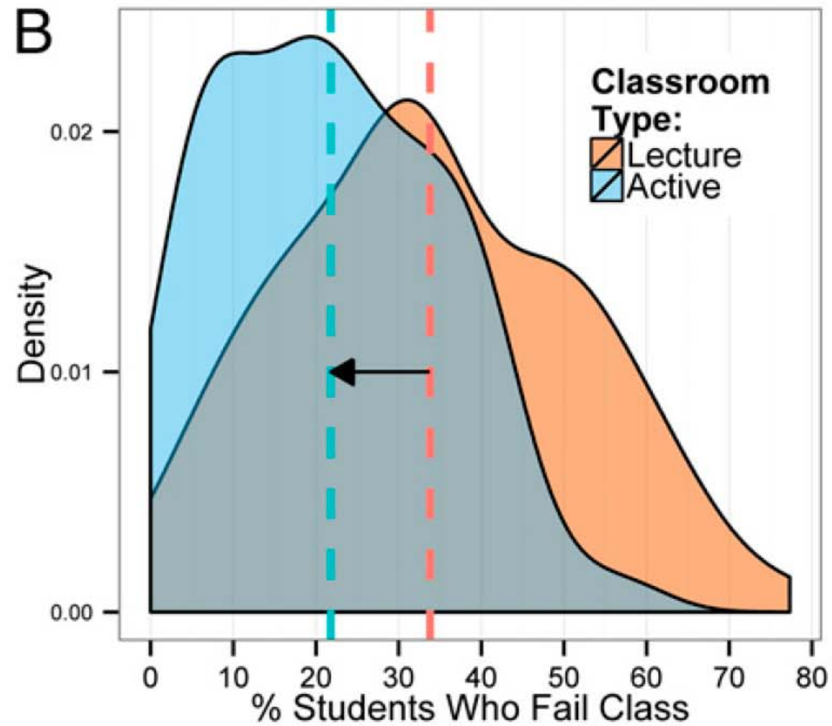
- Meta-analysis of 225 studies comparing student performance in undergraduate science, technology, engineering, and mathematics (44 studies) (STEM) courses under traditional lecturing versus active learning.
- Active learning interventions included:
  - occasional group problem-solving,
  - worksheets or tutorials completed during class,
  - use of personal response systems
  - studio or workshop course designs.
- Research questions:
  - Does active learning boost examination scores?
  - Does it lower failure rates?

Freeman, S. et al, Proceedings of the National Academy of Sciences, June 10, 2014, vol. 111 no. 238410-8414.

Changes in Failure Rate Lecture vs Active Learning



(A) The mean change (12%) is indicated by the dashed vertical line.



(B) Kernel density plots of failure rates under active learning and under lecturing. The mean failure rates under each classroom type (21.8% and 33.8%) are shown by dashed vertical lines.

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Freeman et al., 2014

Active Learning Increases Student Performance in Science, Engineering, and Mathematics

- Active learning confers disproportionate benefits for STEM students from disadvantaged backgrounds and for female students in male-dominated fields;

- The results raise questions about the continued use of traditional lecturing as a control in research studies, and support **active learning as the preferred, empirically validated teaching practice in regular classrooms**.

Freeman et al, 2014

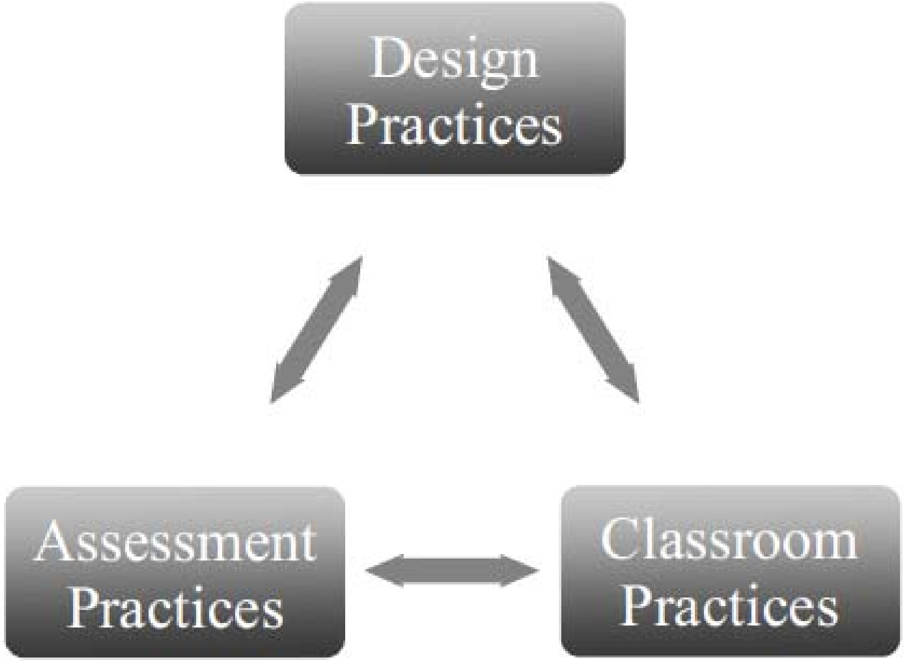
+ **The Conference Board of the Mathematical Sciences (CBMS) Position Statement:**

Active Learning in Post-Secondary Mathematics Education (July, 2016)

". . . *we call on institutions of higher education, mathematics departments and the mathematics faculty, public policy-makers, and funding agencies to invest time and resources to ensure that effective active learning is incorporated into post-secondary mathematics classrooms*."

http://www.cbmsweb.org/2016/07/active-learning-in-post-secondary-mathematics-education/

MAA Instructional Practices Guide



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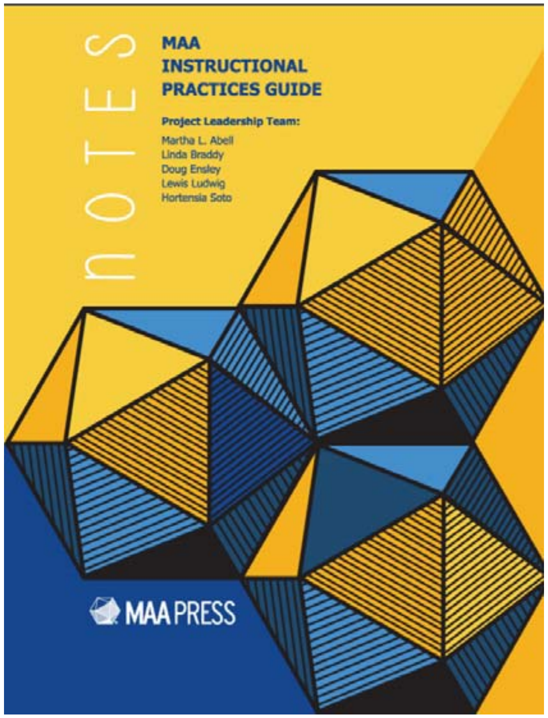
Classroom Practices

Fostering student engagement

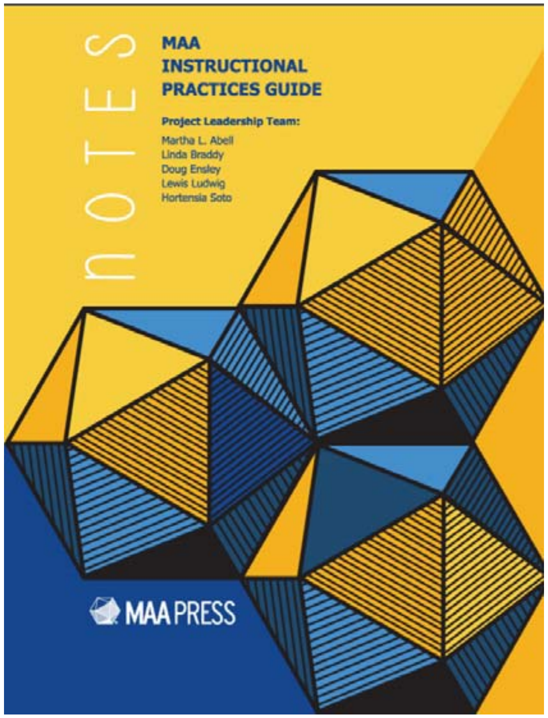
- Building a classroom community
- Wait time
- Responding to student contributions in the classroom
- Exit tickets
- Collaborative learning strategies
- Just-in-time teaching (JiTT)
- Developing persistence in problem solving
- Inquiry-based teaching and learning strategies



Classroom Practices

Selecting appropriate mathematical tasks

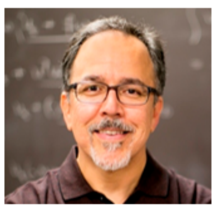
- What makes a mathematical task appropriate?
- How to select an appropriate mathematical task
- Communication: Reading, writing, presenting, visualizing
- Error analysis of student work
- Procedural fluency emerges from conceptual understanding



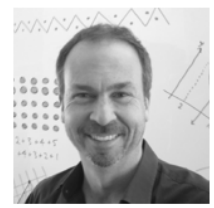
The Mathematical Education of Teachers as an Application of College Mathematics-- META Math

An NSF-funded project to create and field-test materials for use in undergraduate mathematics and statistics courses taken by pre-service teachers.

PIs: James Alvarez, Elizabeth Burroughs, Doug Ensley, Nancy Neudauer, James Tanton







Researcher Group: Elizabeth Arnold, Elizabeth Fulton, Andrew Kercher

Project Evaluator: Cynthia Schneider

Website: sites.google.com/maa.org/metamath/home

MATHEMATICAL ASSOCIATION OF AMERICA

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MONTANA STATE UNIVERSITY

UNIVERSITY OF TEXAS ARLINGTON

JMU JAMES MADISON UNIVERSITY.

NSF

META Math Goals

The project enhances student understanding of the vertical connections from school (K-12) mathematics through advanced undergraduate mathematics among all mathematics undergraduates. All undergraduates, not just future teachers, will benefit from the deep mathematical understanding fostered by instructors using META Math lessons.

Project Objectives:

1. Develop quality instructional lessons that make explicit connections between college mathematics and school mathematics.
   - Discrete Mathematics, Calculus, Abstract Algebra, Statistics
2. Pilot test these lessons.
3. Research student learning that results from the use of these lessons, and evaluate the effectiveness of the annotated lesson plans.

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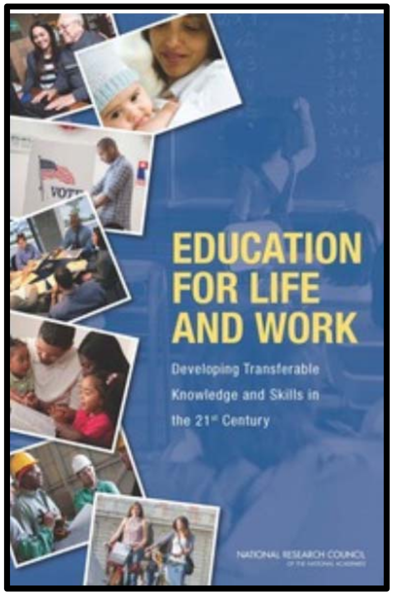
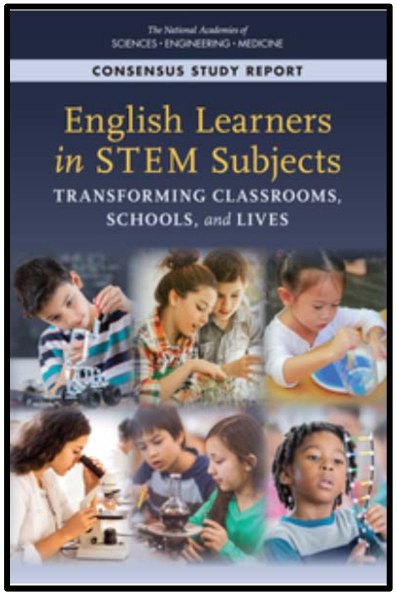
National Research Council/ National Academy Press www.nap.edu

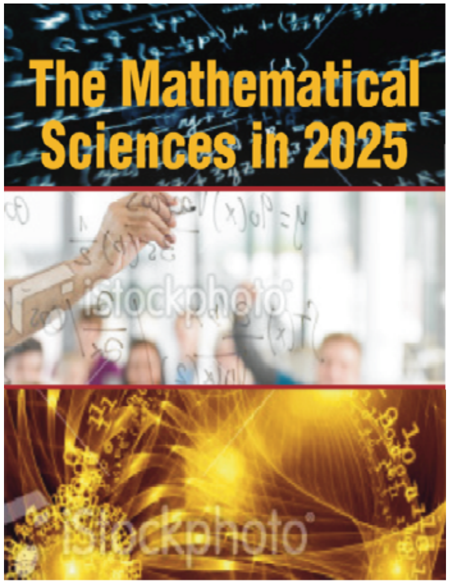
- **Proceedings:**
  Proceedings published by the National Academies of Sciences, Engineering, and Medicine chronicle the presentations and discussions at a workshop, symposium, or other event convened by the National Academies. The statements and opinions contained in proceedings are those of the participants and are not endorsed by other participants, the planning committee, or the National Academies.

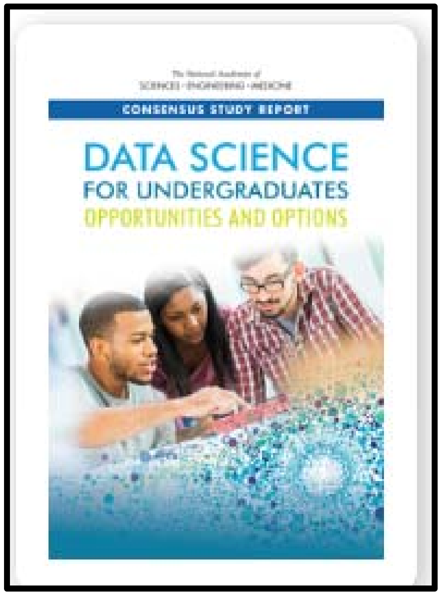
- **Consensus Study Reports:**
  Consensus Study Reports published by the National Academies of Sciences, Engineering, and Medicine document the evidence-based consensus on the study's statement of task by an authoring committee of experts. Reports typically include findings, conclusions, and recommendations based on information gathered by the committee and the committee's deliberations. **Each report has been subjected to a rigorous and independent peer-review process and it represents the position of the National Academies on the statement of task.**

National Research Council/ National Academy Press www.nap.edu

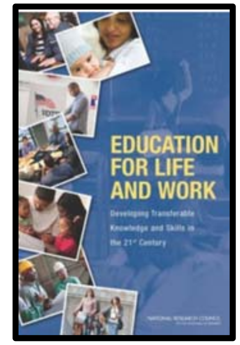








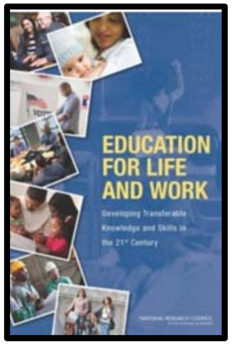
21st Century Competencies



"Deeper learning" is the process through which an individual becomes capable of taking what was learned in one situation and applying it to new situations (i.e., transfer). . . . The product of deeper learning is transferable knowledge, including content knowledge in a domain and knowledge of how, why, and when to apply this knowledge to answer questions and solve problems. We refer to this blend of both knowledge and skills as "21st century competencies."

Education for Life & Work, p. 5

21st Century Competencies



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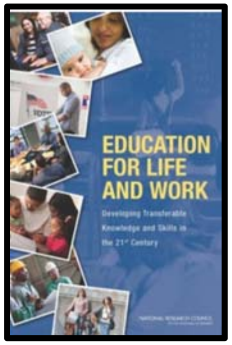
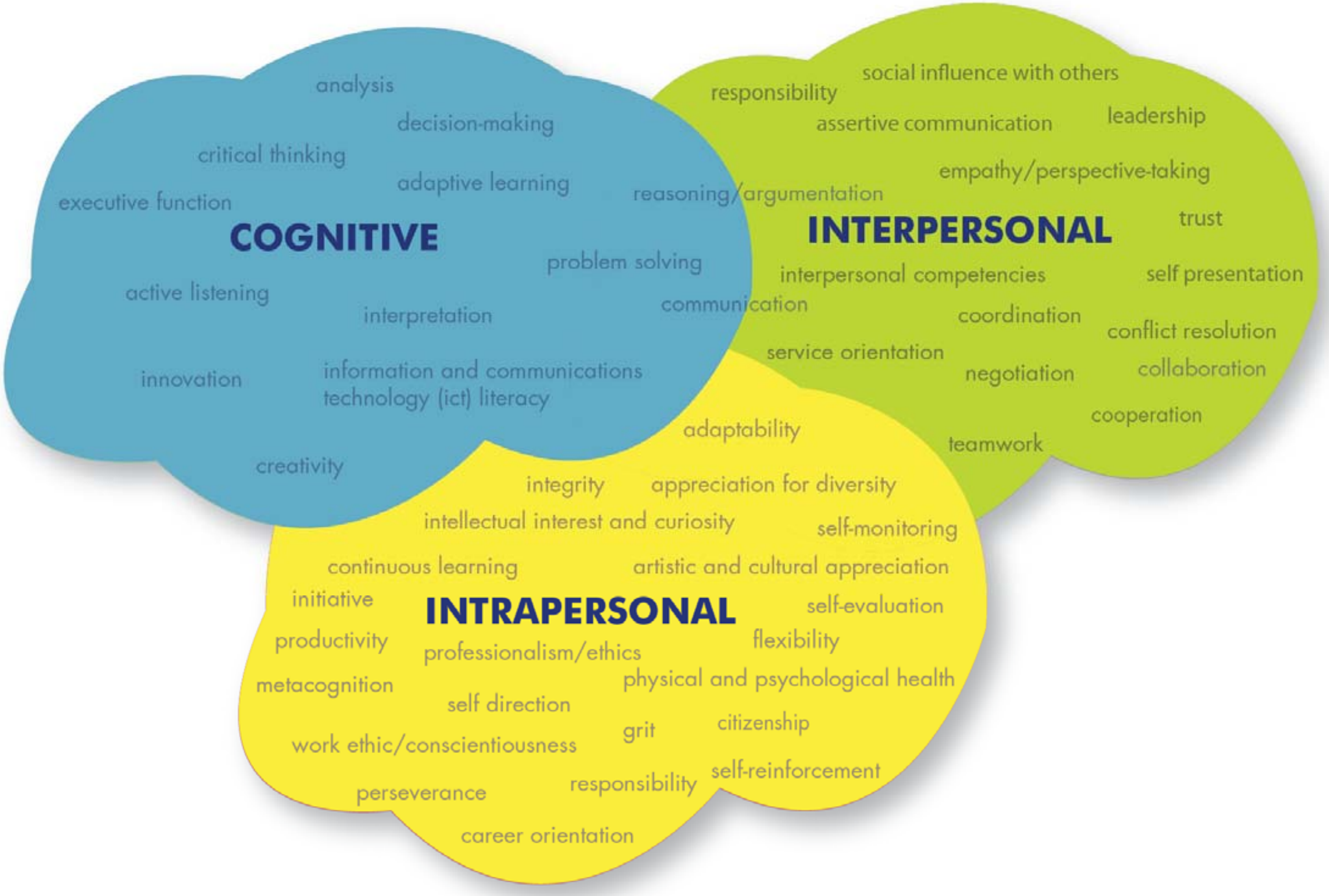
Education for Life & Work, p. 5

21st Century Competencies

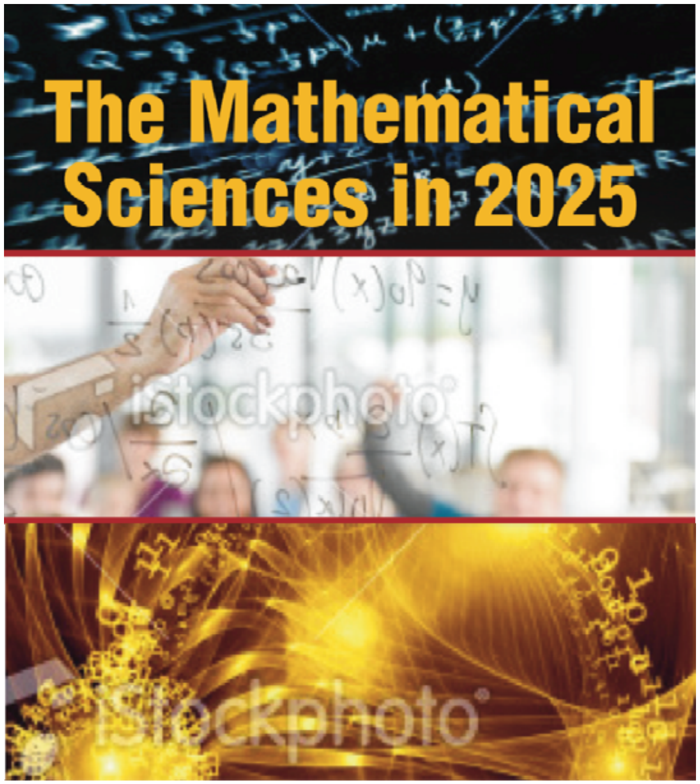




Education for Life & Work, NRC 2012

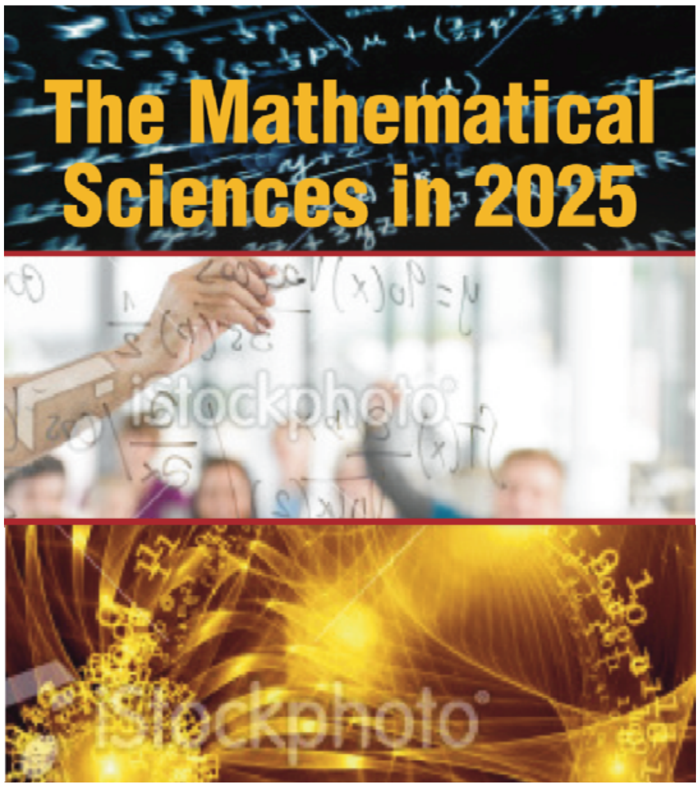
Briars, May 2019

Update Curricula



- Examines the mathematical science now and how it needs to evolve to produce the best value for the country by 2025.
- Describes the remarkable success of the mathematical sciences in the opening years of the 21st century.
- Highlights the increasing importance of statistics, modeling and discrete mathematics.

Update Curricula



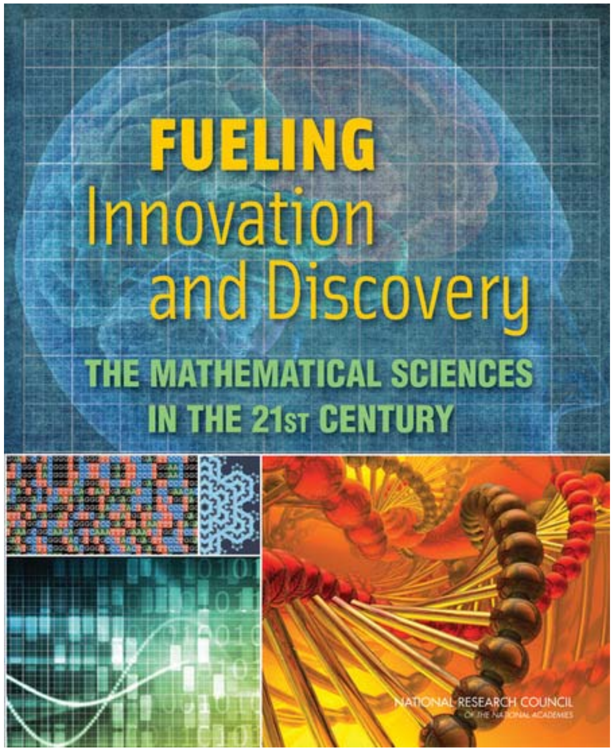
Our high schools focus on getting people prepared for calculus . . .

But we do little to teach statistics, probability, and uncertainty . . .

This is one of the biggest issues facing U.S. mathematical sciences; it is also a big problem in terms of national competitiveness.

Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century

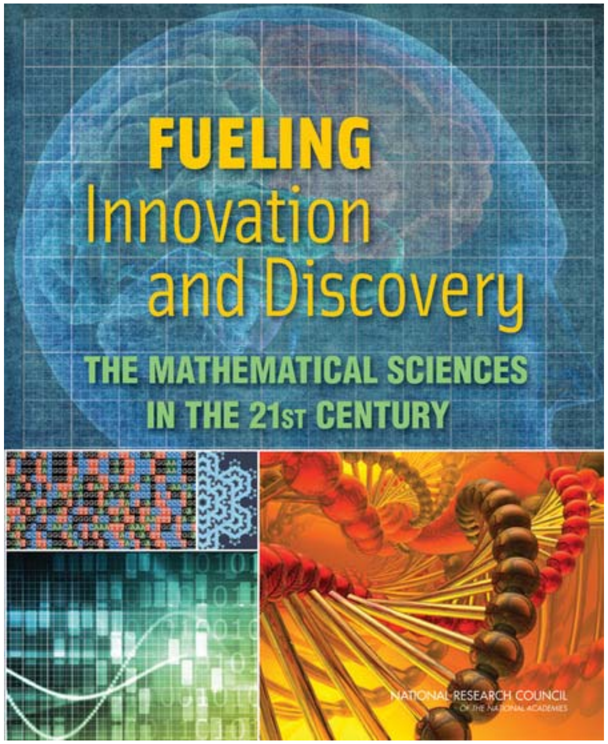
Describes ongoing advances in the mathematical sciences and how these advances are changing our understanding of the world, creating new technologies and transforming industries.



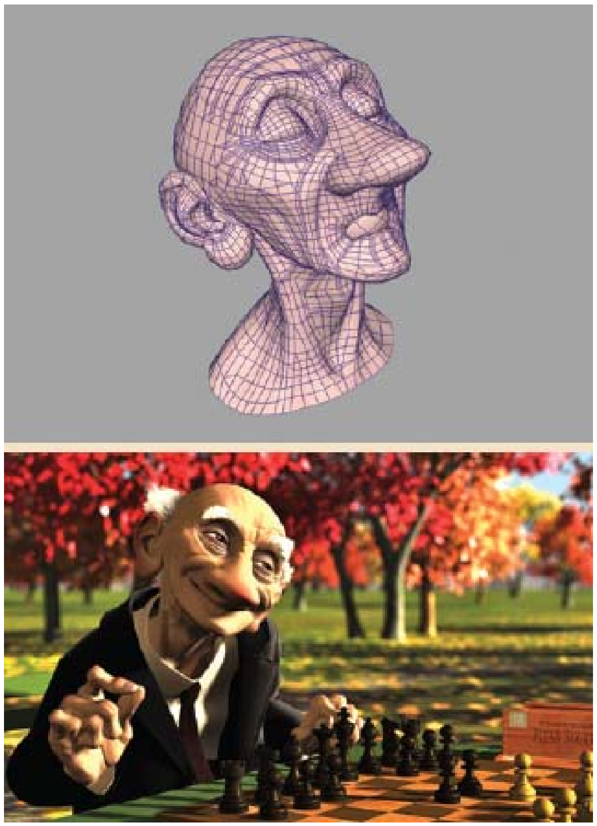
Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century

Purpose

- Communicate the possibilities of applied mathematics
- Interest K-12 students in STEM careers



Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century



Compressed Sensing

Eigenvectors

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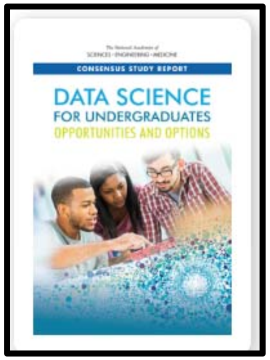

What is Data Science?

Although the definition of data science is evolving, it centers on the notion of multidisciplinary and interdisciplinary approaches to **extracting knowledge or insights from large quantities of complex data for use in a broad range of applications**. Data science is about **synthesizing the most relevant parts of the foundational disciplines to solve particular classes of problems or applications** that are newly enabled because the volume and variety of data available are expanding swiftly, data are available more immediately, and decisions based on data are increasingly automated and in real time. Data scientists often work at the interface of disciplines and can help develop new approaches to address problems in these areas.

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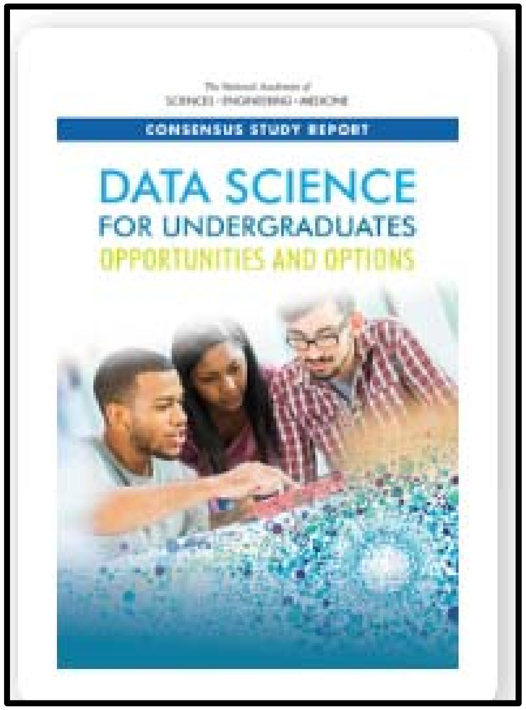
Current Areas of Focus for Data Scientists

- *Computing hardware and software platforms for data science*
- *Data storage and access*
- *Statistical modeling and machine learning*
- *Data visualization*
- *Business analysis*



Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME)

- What is mathematical modeling?
- What does it look like in pK-8, high school, and college?
- Appendices with resources and examples

+ Issues for Teacher Preparation and Professional Learning

Students' Mathematics Identities

Are how students see themselves and how they are seen by others, including teachers, parents, and peers, as doers of mathematics.

Aguirre, Mayfield-Ingram & Martin, 2013

Students' Mathematics Identities

- Mathematics identity includes:
  - beliefs about one's self as a mathematics learner;
  - one's perceptions of how others perceive him or her as a mathematics learner,
  - beliefs about the nature of mathematics,
  - engagement in mathematics, and
  - perception of self as a potential participant in mathematics (Solomon, 2009).

Identity & Motivation

- Understanding the strengths and motivations that serve to develop students' identities should be embedded in the daily work of teachers.
- Mathematics teaching involves not only helping students develop mathematical skills but also empowering students to seeing themselves as being doers of mathematics.

Students' Beliefs about Their Intelligence Effects Their Willingness to Struggle

- **Fixed mindset:**
  - Avoid learning situations if they might make mistakes
  - Try to hide, rather than fix, mistakes or deficiencies
  - Decrease effort when confronted with challenge
- **Growth mindset:**
  - Work to correct mistakes and deficiencies
  - View effort as positive; increase effort when challenged

Dweck, 2007

Students Can Develop Growth Mindsets

- Teacher praise influences mindsets
  - Fixed: Praise refers to intelligence
  - Growth: Praise refers to effort, engagement, perseverance
- Explicit instruction about the brain, its function, and that intellectual development is the result of effort and learning has increased students' achievement in middle school mathematics.
- Reading stories of struggle by successful individuals can promote a growth mindset (Lin et al, 2016)

Messages?

National Alliance for Partnerships in Equity, NAPE, 2016

Micromessages

Small, subtle unconscious messages we send and receive when we interact with others. Negative micromessages cause people to feel devalued, slighted, discouraged or excluded. Positive micromessages cause people to feel valued, included, or encouraged.

National Alliance for Partnerships in Equity, 2016

Micromessages

What are some examples of micromessages that might influence students' mathematics identity?

What Messages Are We Sending About Mathematical Identity?

- “What don’t you understand? This is so simple.”
- “It is immediately obvious that . . . . “
- Which students participate in class?
- How do they participate? What kind of questions are they asked? Wait time? Opportunities for elaboration/explanations?
- Responses to students’ questions? To their answers?
- Feedback: Effort-based vs intelligence-based praise.
- Non-verbal communication—smiles, nods, etc.

Observations to Uncover Micromessages and Unintentional Bias

Collaborate with a colleague to observe the classroom experiences of students--

- Number of interactions
- Amount of wait/think time
- Nature of questions—higher vs lower level
- Nature of feedback
- Amount of eye contact
- Use of language

NPR: The Teacher Who Believes Math Equals Love

Briars, May 2019

NPR: The Teacher Who Believes Math Equals Love

The school had ordered new math textbooks, but she had already decided — as a student teacher — that she wasn't going to use textbooks.

"I don't want to be stifled by that. I mean, I teach a lot of things in a totally different order than a textbook would," she says.

The average math textbook was, itself, a problem to be solved.

"I decided we were gonna make our own textbooks."

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Getting Better Together

Mathematics Lessons that are fun! fun! fun!

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Purplemath

Briars, May 2019

+ Curriculum Recommendations

NCTM Curriculum Principle

An excellent mathematics program includes curriculum that develops important mathematics along coherent learning progressions and develops connections among areas of mathematical study and between mathematics and the real world.

AMTE

[Well-prepared beginning teachers] . . . can read, analyze, and discuss curriculum, assessment, and standards documents as well as students' mathematical productions.

+

Beliefs about the mathematics curriculum and instructional materials

To what extent would your preservice teachers agree or disagree that:

- They should be developing their own lessons?
- That using lessons from a textbook is a sign of a poor teacher?

To what extent do your preservice teachers have the opportunity to analyze instructional materials?

Dividing Fractions - Book 1
Initial Lesson

Investigation 4

Dividing With Fractions

In earlier investigations of this unit, you learned to use addition, subtraction, and multiplication of fractions in a variety of situations. There are times when you also need to divide fractions. To develop ideas about when and how to divide fractions, let's review the meaning of division in problems involving only whole numbers.

Getting Ready for Problem 4.1

Students at Lakeside Middle School raise funds to take a field trip each spring. In each of the following fundraising examples, explain how you recognize what operation(s) to use. Then write a number sentence to show the required calculations.

- The 24 members of the school swim team get dollar-per-mile pledges for a swim marathon they enter. The team goal is to swim 120 miles. How many miles should each swimmer swim?

- There are 360 students going on the field trip. Each school bus carries 30 students. How many buses are needed?
- The school band plans to sell 600 boxes of cookies. There are 20 members in the band. How many boxes should each member sell to reach the goal if each sells the same number of boxes?

Compare your number sentences and reasoning about these problems with classmates. Decide which are correct and why.

4.1 Preparing Food

There are times when the amounts given in a division situation are not whole numbers but fractions. First, you need to understand what division of fractions means. Then you can learn how to calculate quotients when the divisor or the dividend, or both, is a fraction.

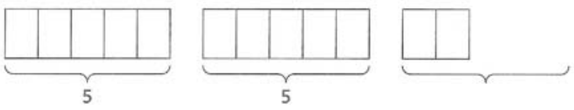
When you do the division $12 \div 5$, what does the answer mean?

The answer should tell you how many fives are in 12 wholes. Because there is not a whole number of fives in 12, you might write:

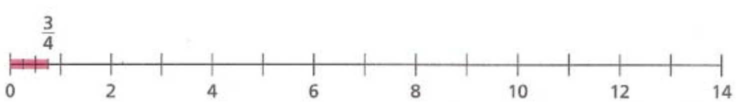
$$12 \div 5 = 2\tfrac{2}{5}$$

Now the question is, what does the *fractional part* of the answer mean?

The answer means you can make 2 fives and $\frac{2}{5}$ *of another five*.



Suppose you ask, "How many $\frac{3}{4}$'s are in 14?" You can write this as a division problem, $14 \div \frac{3}{4}$.



Can you make a whole number of $\frac{3}{4}$'s out of 14 wholes?

If not, what does the fractional part of the answer mean?

As you work through the problems in this investigation, keep these two questions in mind.

What does the answer to a division problem mean?

What does the fractional part of the answer to a division problem mean?

Briars, May 2019

+ Dividing Fractions - Book 1

Initial Lesson

Problem 4.1 Dividing a Whole Number by a Fraction

Use written explanations or diagrams to show your reasoning for each part. Write a number sentence showing your calculation(s).

A. Naylah plans to make small cheese pizzas to sell at a school fundraiser. She has nine bars of cheese. How many pizzas can she make if each pizza needs the given amount of cheese?

1. $\frac{1}{3}$ bar **2.** $\frac{1}{4}$ bar **3.** $\frac{1}{5}$ bar

4. $\frac{1}{6}$ bar **5.** $\frac{1}{7}$ bar **6.** $\frac{1}{8}$ bar

B. Frank also has nine bars of cheese. How many pizzas can he make if each pizza needs the given amount of cheese?

1. $\frac{1}{3}$ bar **2.** $\frac{2}{3}$ bar **3.** $\frac{3}{3}$ bar **4.** $\frac{4}{3}$ bar

5. The answer to part (2) is a mixed number. What does the fractional part of the answer mean?

C. Use what you learned from Questions A and B to complete the following calculations.

1. $12 \div \frac{1}{3}$ **2.** $12 \div \frac{2}{3}$ **3.** $12 \div \frac{5}{3}$

4. $12 \div \frac{1}{6}$ **5.** $12 \div \frac{5}{6}$ **6.** $12 \div \frac{7}{6}$

7. The answer to part (3) is a mixed number. What does the fractional part of the answer mean in the context of cheese pizzas?

D. 1. Explain why $8 \div \frac{1}{3} = 24$ and $8 \div \frac{2}{3} = 12$.

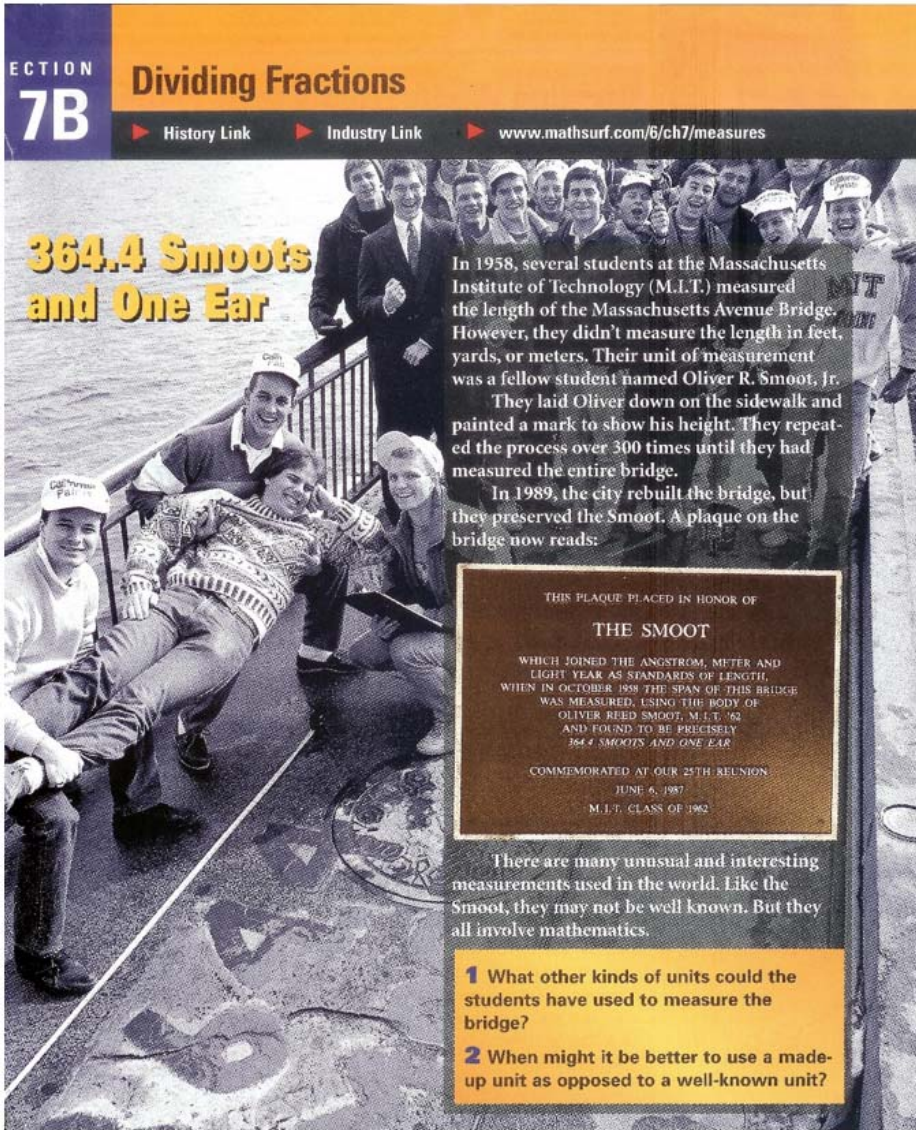
2. Why is the answer to $8 \div \frac{2}{3}$ exactly half the answer to $8 \div \frac{1}{3}$?

E. Write an algorithm that seems to make sense for dividing any whole number by any fraction.

F. Write a story problem that can be solved using $12 \div \frac{2}{3}$. Explain why the calculation matches the story.

+ Dividing Fractions - Book 2

Initial Lesson



ECTION 7B

Dividing Fractions

History Link · Industry Link · www.mathsurf.com/6/ch7/measures

364.4 Smoots and One Ear

In 1958, several students at the Massachusetts Institute of Technology (M.I.T.) measured the length of the Massachusetts Avenue Bridge. However, they didn't measure the length in feet, yards, or meters. Their unit of measurement was a fellow student named Oliver R. Smoot, Jr.

They laid Oliver down on the sidewalk and painted a mark to show his height. They repeated the process over 300 times until they had measured the entire bridge.

In 1989, the city rebuilt the bridge, but they preserved the Smoot. A plaque on the bridge now reads:

THIS PLAQUE PLACED IN HONOR OF

THE SMOOT

WHICH JOINED THE ANGSTROM, METER AND LIGHT YEAR AS STANDARDS OF LENGTH, WHEN IN OCTOBER 1958 THE SPAN OF THIS BRIDGE WAS MEASURED, USING THE BODY OF OLIVER REED SMOOT, M.I.T. '62 AND FOUND TO BE PRECISELY *364.4 SMOOTS AND ONE EAR*

COMMEMORATED AT OUR 25TH REUNION

JUNE 6, 1987

M.I.T. CLASS OF 1962

There are many unusual and interesting measurements used in the world. Like the Smoot, they may not be well known. But they all involve mathematics.

1 What other kinds of units could the students have used to measure the bridge?

2 When might it be better to use a made-up unit as opposed to a well-known unit?

Briars, May 2019

Dividing Fractions - Book 2

Initial Lesson

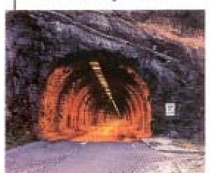
7-4 Dividing Whole Numbers by Fractions

You'll Learn ...

- to divide a whole number by a fraction

... How It's Used

Structural engineers divide whole numbers by fractions when building tunnels.



Vocabulary

reciprocal

▶ Lesson Link In the last section, you learned to multiply whole numbers by fractions. Now you'll divide whole numbers by fractions. ◀

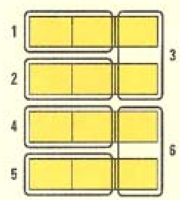
Explore Dividing Whole Numbers by Fractions

Circles and Strips Forever

Dividing a Whole Number by a Fraction

- Draw a number of strips equal to the whole number.
- Divide the strips into equal pieces. The number of pieces in each strip should be equal to the fraction denominator.
- Circle groups of equal pieces. The number of pieces in each circled group should equal the numerator.
- Describe the number of groups circled.

$4 \div \frac{2}{3} = 6$



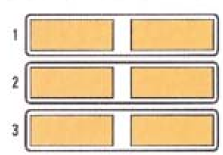
1. Model these problems.

   a. $6 \div \frac{2}{3}$ **b.** $7 \div \frac{1}{2}$ **c.** $5 \div \frac{5}{6}$ **d.** $4 \div \frac{3}{6}$ **e.** $2 \div \frac{2}{7}$

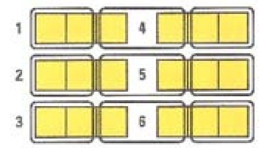
2. When you divide a whole number by a fraction less than 1, is the quotient larger or smaller than the original whole number? Why?
3. Will $3 \div \frac{2}{5}$ have a whole-number answer? Explain.

Learn Dividing Whole Numbers by Fractions

You can think of division as taking a given amount and breaking it down into groups of a certain size. For example, $6 \div 2$ can be modeled as 6 loaves of bread divided into groups of 2. The quotient, 3, is the number of groups you have.



You can think of dividing by fractions in the same way. For example, $6 \div \frac{2}{3}$ is the same as 6 loaves of bread divided into groups of $\frac{2}{3}$. The number of groups you have, 9, is the quotient.



Notice that to find the answer, you first found the number of thirds by multiplying the number of loaves, 6, by the denominator, 3. Then, you divided the number of thirds by the numerator, 2.

$6 \div \frac{2}{3} = 6 \times 3 \div 2 = 9$

Dividing by a fraction is the same as multiplying by its **reciprocal**. Reciprocals are numbers whose numerators and denominators have been switched. When two numbers are reciprocals, their product is 1.

Dividing	Multiplying by reciprocal
$6 \div \frac{2}{3} = 9$	$6 \times \frac{3}{2} = \frac{6}{1} \times \frac{3}{2} = \frac{18}{2} = 9$

Remember

The numerator is the number on top of a fraction. The denominator is the number on the bottom. [Page 287]

Examples

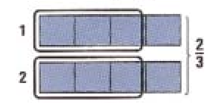
1 Divide: $2 \div \frac{3}{4}$

$2 \div \frac{3}{4} = \frac{2}{1} \times \frac{4}{3}$ Multiply by the reciprocal of the fraction.

$= \frac{2 \times 4}{1 \times 3}$

$= \frac{8}{3}$ or $2\frac{2}{3}$

$2 \div \frac{3}{4} = 2\frac{2}{3}$



2 1 *nail* = $\frac{9}{4}$ in. of cloth. Find the length of 5 in. of cloth in nails.

$5 \div \frac{9}{4} = \frac{5}{1} \times \frac{4}{9}$ Multiply by the reciprocal.

$= \frac{20}{9}$ or $2\frac{2}{9}$ Simplify.

A 5-inch piece of cloth is $2\frac{2}{9}$ nails long.

Try It

Divide. **a.** $4 \div \frac{3}{5}$ **b.** $1 \div \frac{4}{7}$ **c.** $10 \div \frac{17}{4}$ **d.** $3 \div \frac{3}{5}$

DID YOU KNOW?

Three measurements used primarily for cloth include the *nail*, the *finger*, and the *span*. A *finger* is equal to $4\frac{1}{2}$ inches. A *span* is equal to 9 inches.

Dividing Fractions - Book 2

Initial Lesson

WHAT DO YOU THINK?

Peter and Erica have a recipe that makes 9 quarts of punch. They want to know how many $\frac{3}{4}$-quart (3-cup) servings the recipe will make.

Peter thinks ...

I'll use mental math.

How many groups of $\frac{3}{4}$ quart are in 9 quarts?

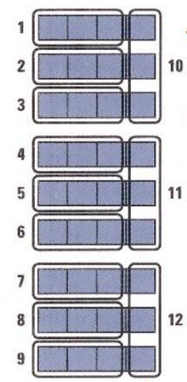
Every whole quart has one $\frac{3}{4}$ in it, plus $\frac{1}{4}$ left over.

In 9 quarts, there are nine $\frac{3}{4}$ quarts, which is 9 servings.

There are nine $\frac{1}{4}$ quarts left over. They can be regrouped as three $\frac{3}{4}$ quarts, which is another 3 servings.

That equals 9 + 3 or 12 groups of $\frac{3}{4}$ quart.

We will have 12 servings.



Erica thinks ...

I'll divide 9 by $\frac{3}{4}$. To do that, I'll multiply $\frac{9}{1}$ by the reciprocal of $\frac{3}{4}$.

$$9 \div \frac{3}{4} = \frac{9}{1} \times \frac{4}{3} = 12$$

We will have 12 servings.

What do you think?

1. Whose method is easier to use without paper and pencil? Explain.
2. How could you find the answer by writing 9 quarts as $\frac{36}{4}$ quarts?

Check Your Understanding

1. How could you use the "multiply by the reciprocal" rule to divide 20 by 5?
2. If you divide a whole number by a proper fraction, is the quotient larger or smaller than the whole number? Explain.

Dividing Fractions - Book 2

Initial Lesson--Homework

7-4 Exercises and Applications

Practice and Apply

Getting Started State the reciprocal.

1. $\frac{5}{7}$ **2.** $\frac{1}{2}$ **3.** $\frac{2}{9}$ **4.** $\frac{10}{14}$ **5.** $\frac{1}{4}$ **6.** $\frac{4}{5}$

Simplify.

7. $6 \div \frac{2}{3}$ **8.** $2 \div \frac{3}{5}$ **9.** $3 \div \frac{6}{7}$ **10.** $1 \div 1\frac{1}{2}$

11. $9 \div \frac{4}{5}$ **12.** $7 \div \frac{6}{5}$ **13.** $4 \div 3\frac{5}{8}$ **14.** $5 \div \frac{1}{4}$

15. $10 \div 7\frac{2}{3}$ **16.** $8 \div 8\frac{7}{8}$ **17.** $3 \div \frac{10}{11}$ **18.** $5 \div \frac{9}{2}$

19. $16 \div \frac{2}{5}$ **20.** $7 \div 6\frac{3}{4}$ **21.** $8 \div 2\frac{1}{6}$ **22.** $2 \div 4\frac{2}{7}$

23. $1 \div 3\frac{5}{9}$ **24.** $4 \div 1\frac{1}{2}$ **25.** $9 \div \frac{6}{7}$ **26.** $6 \div \frac{8}{12}$

27. $11 \div \frac{13}{2}$ **28.** $10 \div 9\frac{8}{9}$ **29.** $3 \div 11\frac{1}{3}$ **30.** $7 \div 2\frac{3}{8}$

31. **Test Prep** Which two expressions have the same quotient as $6 \div 1\frac{3}{4}$?

I. $\frac{6}{1} \div \frac{7}{4}$ II. $\frac{6}{1} \times \frac{7}{4}$ III. $\frac{6}{1} \div \frac{4}{7}$ IV. $\frac{6}{1} \times \frac{4}{7}$

Ⓐ I and II Ⓑ I and IV

Ⓒ III and II Ⓓ III and IV

32. **Science** $\frac{4}{5}$ of a cubic foot of copper weighs 440 pounds. What is the weight of 1 cubic foot of copper?

33. **Social Studies** As a result of the 1990 census, Pennsylvania has 21 seats in the House of Representatives. This is $\frac{7}{10}$ as many seats as Texas has. How many seats does Texas have?

Problem Solving and Reasoning

34. Journal Explain how you can tell if two numbers are reciprocals of each other.

35. **Critical Thinking** This recipe makes 1 batch of cookies. About how many batches can you make if you change the recipe to include the following? Explain your answers.

a. A 2-pound bag of flour? (1 cup = $\frac{1}{4}$ pound)

b. A pound of margarine? (1 cup = $\frac{1}{2}$ pound)

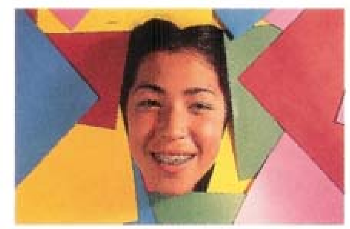
c. A 4-pound bag of white sugar? (1 cup = $\frac{1}{2}$ pound)

Chocolate Chip Cookies

$2\frac{1}{4}$ cups flour
1 teaspoon baking soda
1 teaspoon salt
1 cup margarine
$\frac{3}{4}$ cup white sugar
$\frac{1}{4}$ cup packed brown sugar
1 teaspoon vanilla extract
2 eggs
2 cups chocolate chips

36. **Communicate** Is $4 \div \frac{2}{5}$ the same as $\frac{2}{5} \div 4$? Explain your reasoning.

37. **Critical Thinking** A *ream* of paper is 500 sheets. A *quire* of paper is $\frac{1}{20}$ of a ream. Monique wanted to know how many sheets of paper were in a quire. She calculated $500 \div \frac{1}{20} = 10{,}000$, and decided a quire of paper was 10,000 sheets. Is her answer reasonable? Explain.



Mixed Review

Convert. *[Lesson 4-3]*

38. 144 ounces = ☐ pounds **39.** 56 pounds = ☐ ounces **40.** 80 ounces = ☐ pounds

41. 100 gallons = ☐ quarts **42.** 64 quarts = ☐ gallons **43.** 40 gallons = ☐ quarts

For each fraction, draw a model. *[Lesson 5-4]*

44. $\frac{1}{4}$ **45.** $\frac{7}{8}$ **46.** $\frac{4}{7}$ **47.** $\frac{80}{100}$ **48.** $\frac{9}{15}$ **49.** $\frac{5}{7}$

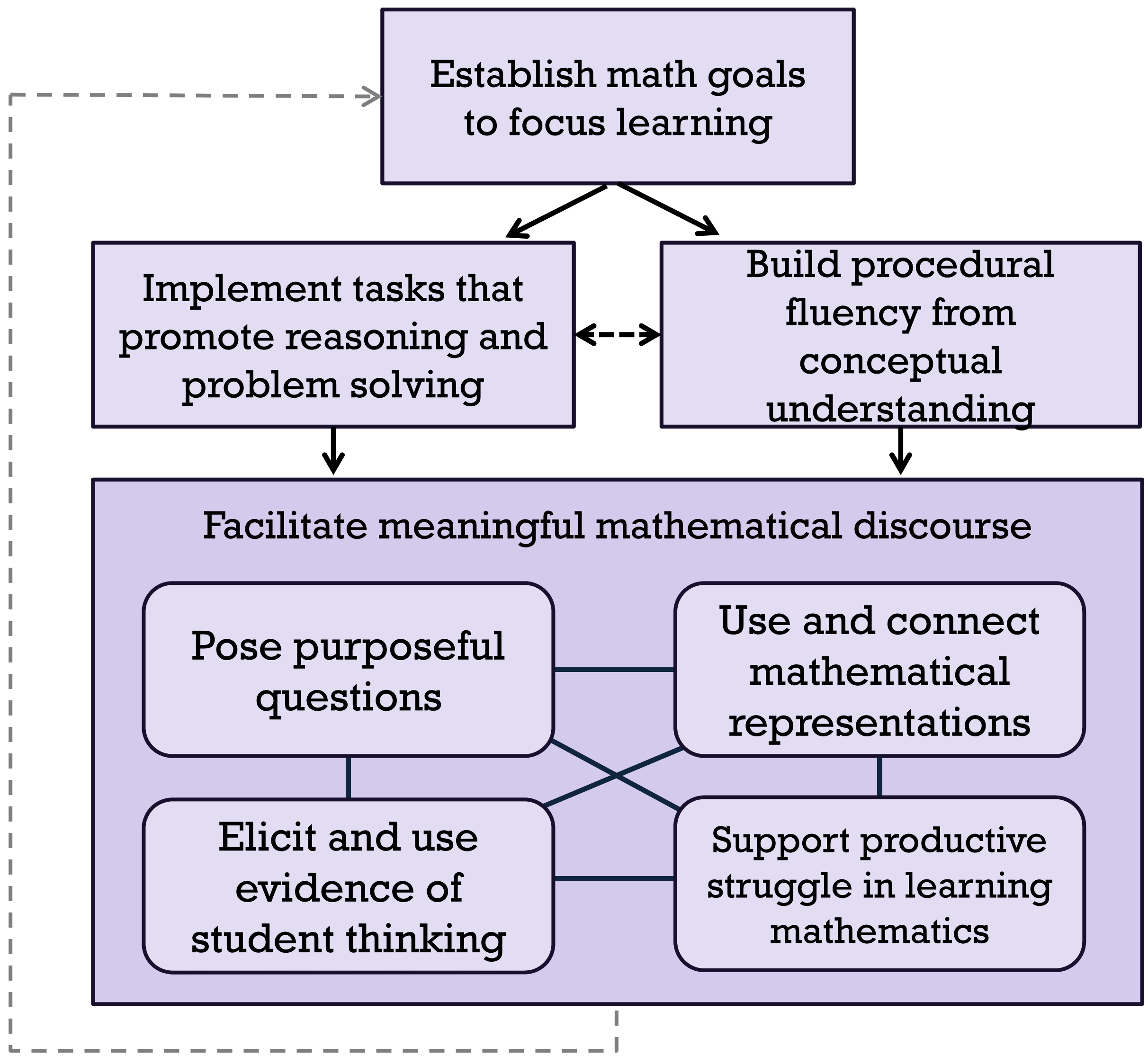
Project Progress

Choose 10 of the items from your list. Make a chart detailing how much each item cost when your senior citizen was your age, and how much it costs today. Estimate the fraction or mixed number you would need to multiply the old price by to get the current price.

Problem Solving

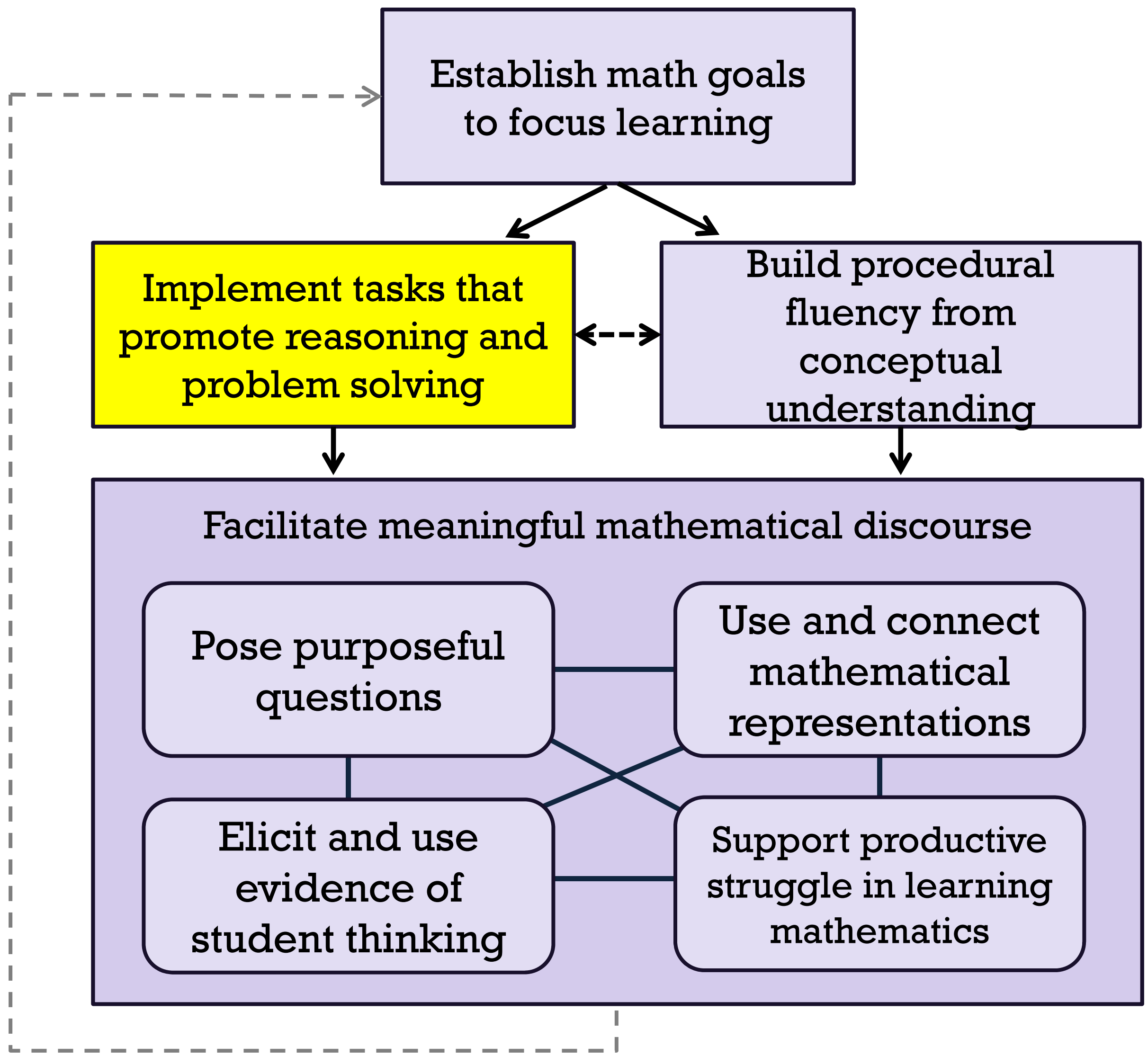
Understand
Plan
Solve
Look Back

Effective Mathematics Teaching Practices



Taking Action: Implementing Effective Teaching Practices, NCTM, 2017

Effective Mathematics Teaching Practices



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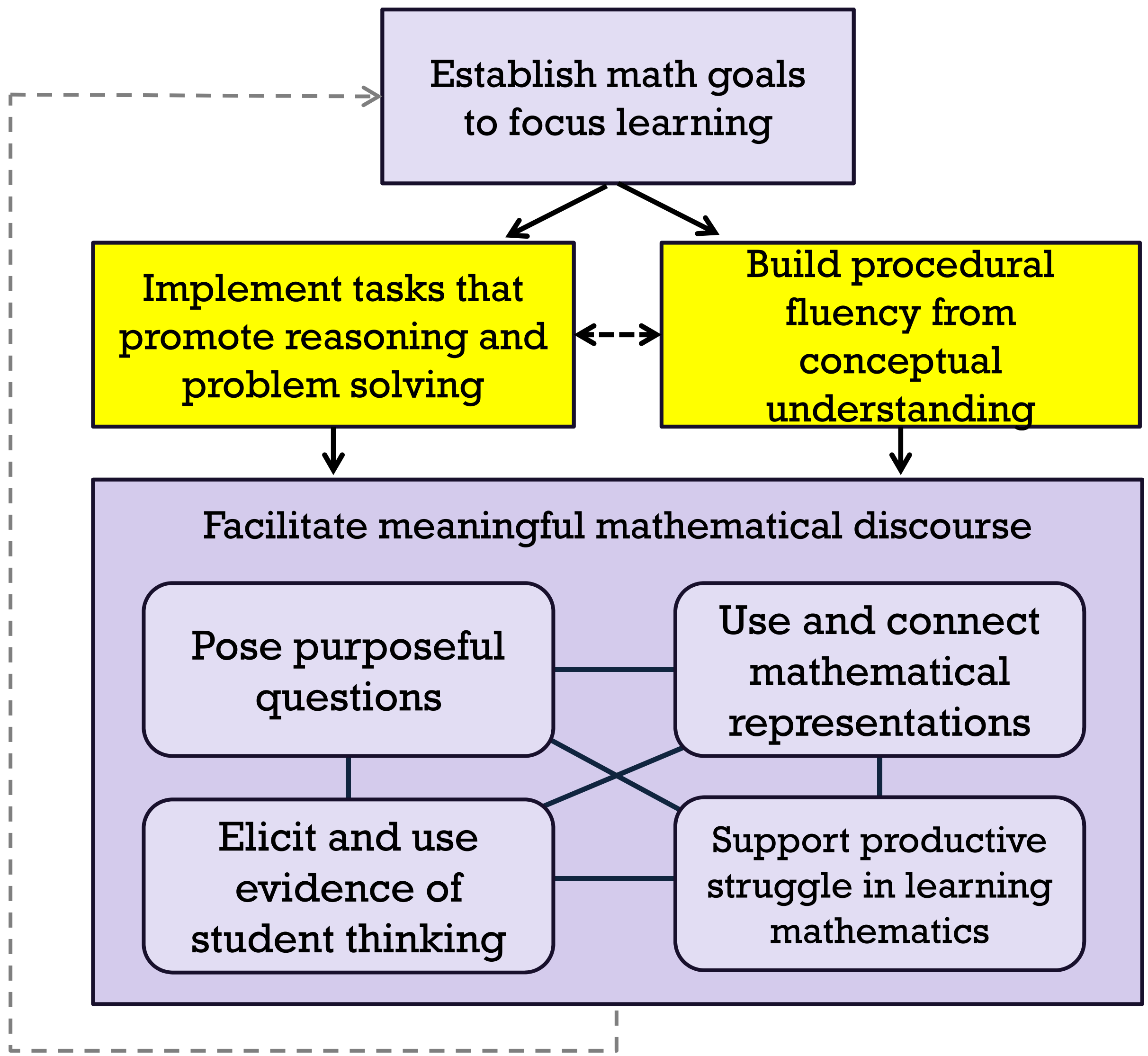
Effective Mathematics Teaching Practices



Taking Action: Implementing Effective Teaching Practices, NCTM, 2017

Dividing Fractions—Book 1

Problem 4.1 Dividing a Whole Number by a Fraction

Use written explanations or diagrams to show your reasoning for each part. Write a number sentence showing your calculatiòn(s).

A. Naylah plans to make small cheese pizzas to sell at a school fundraiser. She has nine bars of cheese. How many pizzas can she make if each pizza needs the given amount of cheese?

1. $\frac{1}{3}$ bar **2.** $\frac{1}{4}$ bar **3.** $\frac{1}{5}$ bar

4. $\frac{1}{6}$ bar **5.** $\frac{1}{7}$ bar **6.** $\frac{1}{8}$ bar

B. Frank also has nine bars of cheese. How many pizzas can he make if each pizza needs the given amount of cheese?

1. $\frac{1}{3}$ bar **2.** $\frac{2}{3}$ bar **3.** $\frac{3}{3}$ bar **4.** $\frac{4}{3}$ bar

5. The answer to part (2) is a mixed number. What does the fractional part of the answer mean?

C. Use what you learned from Questions A and B to complete the following calculations.

1. $12 \div \frac{1}{3}$ **2.** $12 \div \frac{2}{3}$ **3.** $12 \div \frac{5}{3}$

4. $12 \div \frac{1}{6}$ **5.** $12 \div \frac{5}{6}$ **6.** $12 \div \frac{7}{6}$

7. The answer to part (3) is a mixed number. What does the fractional part of the answer mean in the context of cheese pizzas?

D. **1.** Explain why $8 \div \frac{1}{3} = 24$ and $8 \div \frac{2}{3} = 12$.

2. Why is the answer to $8 \div \frac{2}{3}$ exactly half the answer to $8 \div \frac{1}{3}$?

E. Write an algorithm that seems to make sense for dividing any whole number by any fraction.

F. Write a story problem that can be solved using $12 \div \frac{2}{3}$. Explain why the calculation matches the story.

+ Dividing Fractions—Book 2

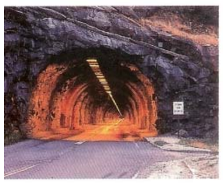
7-4 Dividing Whole Numbers by Fractions

You'll Learn ...

- to divide a whole number by a fraction

... How It's Used

Structural engineers divide whole numbers by fractions when building tunnels.



Vocabulary

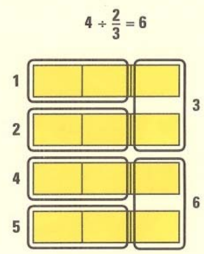
reciprocal

Lesson Link In the last section, you learned to multiply whole numbers by fractions. Now you'll divide whole numbers by fractions.

Explore Dividing Whole Numbers by Fractions

Circles and Strips Forever

Dividing a Whole Number by a Fraction



- Draw a number of strips equal to the whole number.
- Divide the strips into equal pieces. The number of pieces in each strip should be equal to the fraction denominator.
- Circle groups of equal pieces. The number of pieces in each circled group should equal the numerator.
- Describe the number of groups circled.

1. Model these problems.

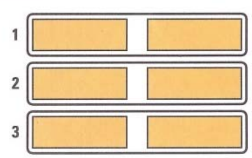
a. $6 \div \frac{2}{3}$ **b.** $7 \div \frac{1}{2}$ **c.** $5 \div \frac{5}{6}$ **d.** $4 \div \frac{3}{6}$ **e.** $2 \div \frac{2}{7}$

2. When you divide a whole number by a fraction less than 1, is the quotient larger or smaller than the original whole number? Why?

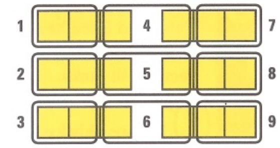
3. Will $3 \div \frac{2}{5}$ have a whole-number answer? Explain.

Learn Dividing Whole Numbers by Fractions

You can think of division as taking a given amount and breaking it down into groups of a certain size. For example, $6 \div 2$ can be modeled as 6 loaves of bread divided into groups of 2. The quotient, 3, is the number of groups you have.



You can think of dividing by fractions in the same way. For example, $6 \div \frac{2}{3}$ is the same as 6 loaves of bread divided into groups of $\frac{2}{3}$. The number of groups you have, 9, is the quotient.



Notice that to find the answer, you first found the number of thirds by multiplying the number of loaves, 6, by the denominator, 3. Then, you divided the number of thirds by the numerator, 2.

$6 \div \frac{2}{3} = 6 \times 3 \div 2 = 9$

Dividing by a fraction is the same as multiplying by its **reciprocal**. Reciprocals are numbers whose numerators and denominators have been switched. When two numbers are reciprocals, their product is 1.

Dividing	Multiplying by reciprocal
$6 \div \frac{2}{3} = 9$	$6 \times \frac{3}{2} = \frac{6}{1} \times \frac{3}{2}$
	$= \frac{18}{2}$
	$= 9$

Remember

The numerator is the number on top of a fraction. The denominator is the number on the bottom. [Page 287]

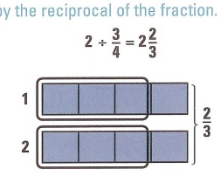
Examples

1 Divide: $2 \div \frac{3}{4}$

$2 \div \frac{3}{4} = \frac{2}{1} \times \frac{4}{3}$ Multiply by the reciprocal of the fraction.

$= \frac{2 \times 4}{1 \times 3}$

$= \frac{8}{3}$ or $2\frac{2}{3}$



2 1 *nail* = $\frac{9}{4}$ in. of cloth. Find the length of 5 in. of cloth in nails.

$5 \div \frac{9}{4} = \frac{5}{1} \times \frac{4}{9}$ Multiply by the reciprocal.

$= \frac{20}{9}$ or $2\frac{2}{9}$ Simplify.

A 5-inch piece of cloth is $2\frac{2}{9}$ nails long.

Try It

Divide. **a.** $4 \div \frac{3}{5}$ **b.** $1 \div \frac{4}{7}$ **c.** $10 \div \frac{17}{4}$ **d.** $3 \div \frac{3}{5}$

DID YOU KNOW?

Three measurements used primarily for cloth include the *nail*, the *finger*, and the *span*. A *finger* is equal to $4\frac{1}{2}$ inches. A *span* is equal to 9 inches.

Briars, May 2019

Dividing Fractions—Standard Algorithm

4.4 Writing a Division Algorithm

You are ready now to develop an algorithm for dividing fractions. To get started, you will break division problems into categories and write steps for each kind of problem. Then you can see whether there is one "big" algorithm that will solve them all.

Problem 4.4 Writing a Division Algorithm

A. 1. Find the quotients in each group below.

Group 1	Group 2	Group 3	Group 4
$\frac{1}{3} \div 9$	$12 \div \frac{1}{6}$	$\frac{5}{6} \div \frac{1}{12}$	$5 \div 1\frac{1}{2}$
$\frac{1}{6} \div 12$	$5 \div \frac{2}{3}$	$\frac{3}{4} \div \frac{3}{4}$	$\frac{1}{2} \div 3\frac{2}{3}$
$\frac{3}{5} \div 6$	$3 \div \frac{2}{5}$	$\frac{9}{5} \div \frac{1}{2}$	$3\frac{1}{3} \div \frac{2}{3}$

2. Describe what the problems in each group have in common.

3. Make up one new problem that fits in each group.

4. Write an algorithm that works for dividing *any* two fractions, including mixed numbers. Test your algorithm on the problems in the table. If necessary, change your algorithm until you think it will work all the time.

B. Use your algorithm to divide.

1. $9 \div \frac{4}{5}$ **2.** $1\frac{7}{8} \div 3$ **3.** $1\frac{2}{3} \div \frac{1}{5}$ **4.** $2\frac{5}{6} \div 1\frac{1}{3}$

C. Here is a multiplication-division fact family for whole numbers:

$5 \times 8 = 40$ $8 \times 5 = 40$ $40 \div 5 = 8$ $40 \div 8 = 5$

1. Complete this multiplication-division fact family for fractions.

$$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

2. Check the division answers by using your algorithm.

D. For each number sentence, find a value for N that makes the sentence true. If needed, use fact families.

1. $\frac{2}{3} \div \frac{4}{5} = N$ **2.** $\frac{3}{4} \div N = \frac{7}{8}$ **3.** $N \div \frac{1}{4} = 3$

Dividing Fractions by Fractions

Lesson Link In the last lesson, you learned to divide whole numbers by fractions. Now you'll divide fractions by fractions.

Explore Dividing Fractions by Fractions

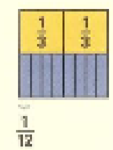
Wish Upon a Bar

Materials: Fraction Bars®

Dividing a Fraction by a Fraction

- Using a Fraction Bar®, draw and label the first fraction.
- Under that, use a Fraction Bar® to draw as many diagrams of the second fraction as will fit.
- Describe the number of diagrams below the first fraction.

$\frac{2}{3} \div \frac{1}{12} = 8$



1. Model each problem.

a. $\frac{3}{6} \div \frac{1}{12}$ **b.** $\frac{1}{2} \div \frac{1}{4}$ **c.** $\frac{2}{3} \div \frac{1}{6}$ **d.** $\frac{2}{4} \div \frac{2}{12}$

2. When you divide a fraction by a fraction less than 1, why is the answer bigger than the fraction you started with?

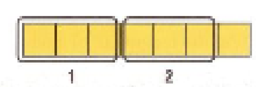
3. How is dividing a fraction by a fraction similar to dividing a whole number by a fraction?

4. Can you use Fraction Bars® to divide $\frac{1}{2} \div \frac{1}{5}$? Explain.

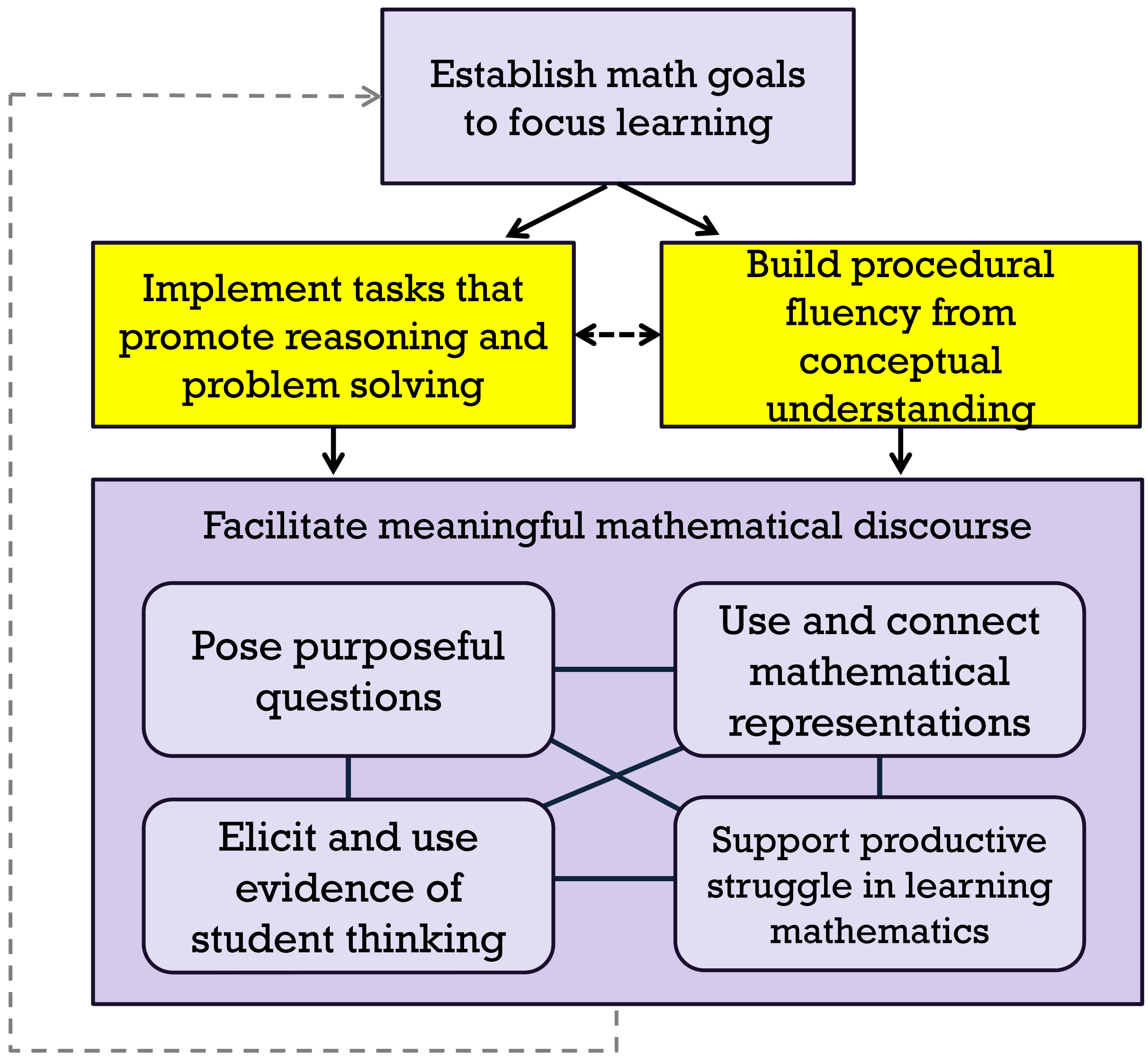
Learn Dividing Fractions by Fractions

When you divide a whole number by a fraction, you get the same result as if you had multiplied the whole number by the fraction's reciprocal. This is also true when you divide a fraction by a fraction.

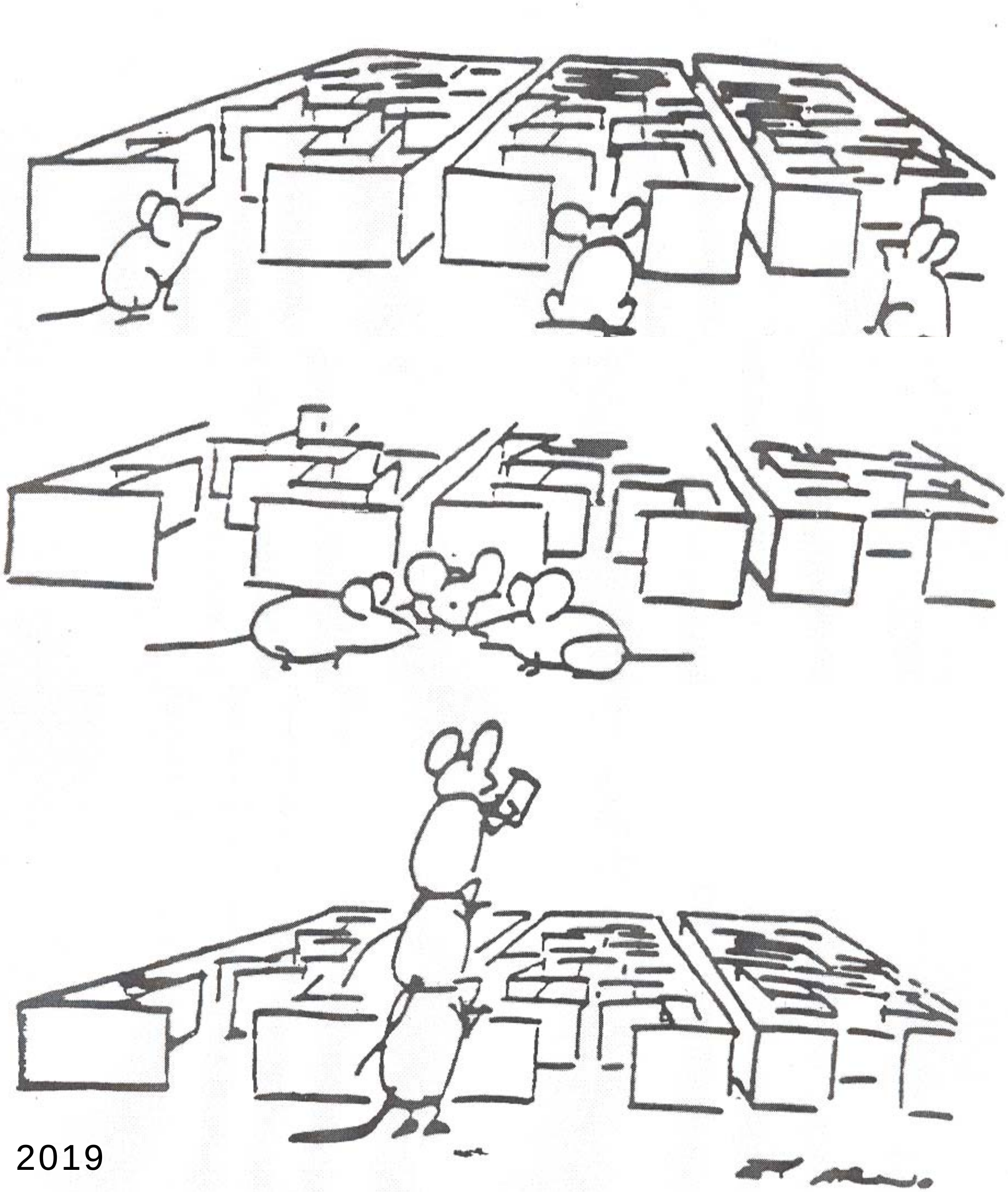
Dividing	Multiplying by Reciprocal
$\frac{6}{7} \div \frac{3}{7} = 2$	$\frac{6}{7} \times \frac{7}{3} = \frac{42}{21}$ or 2



Effective Mathematics Teaching Practices



Taking Action: Implementing Effective Teaching Practices, NCTM, 2017



Briars, May 2019

Professionalism

NCTM

In an excellent mathematics program, educators hold themselves and their colleagues accountable for the mathematical success of every student and for their personal and **collective professional growth** toward effective teaching and learning of mathematics.

AMTE

Well-prepared beginning teachers of mathematics understand the importance of being a part of a community of educators and believe that the community has the potential to affect teaching in a positive way.

Briars, May 2019

What About Assessment?

2025 NAEP Mathematics Framework Update

Issues for Updating the Mathematics Framework

1. How should NAEP assess what students are learning?
2. How should NAEP assess problem-solving?
3. How should NAEP take better advantage of digitally-based assessments?
4. What guidelines will help to make NAEP as fair as possible?
5. What new research should be cited and incorporated?

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What About Assessment?

2025 NAEP Mathematics Framework Update

1. DOWNLOAD the draft Framework
   http://bit.ly/NAEP-DownloadPage
2. Add comments and/or create a separate file with your suggestions.
3. Upload your comment file through a special link; the link is in the file you download.

The upload link closes on **June 7, 2019**.

Themes

- Policies and practices that support success of each and every student (equity)
- Need to update the mathematics curriculum
  - Modeling
  - Statistics
  - Mathematical processes/SMPs
- Effective mathematics instructional practices

Thank You!

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